UMASS AMHERST MATH 300 FALL '05, F. HAJIR

HOMEWORK 3: SETS AND MAPS

HW 3 is due in class on Thursday October 6.

NOTE THAT I AM GIVING YOU TWO (2) WEEKS FOR COMPLETING THIS ASSIGNMENT. IT IS RATHER LONG ... DO NOT WAIT UNTIL THE LAST MINTUE TO START WORK ON IT!

1. Reading

Please read 6.1, 6.3, 6.4, 6.5, 6.6 of Gilbert/Vanstone. You should also read Part III of Farshid's notes.

2. Problems from Gilbert/Vanstone

Exc. Set 6 (p. 152): 1-8, 15-17, 24-28,34,38,43.

Problem Set 6 (p. 157): 94, 99, 100.

3. Problems from Farshid's Brain

1. Write down the negation of the following statements; in each case attempt to cast the statement in positive terms, meaning attempt to eliminate the word "not" from your statement.

For example, it is correct to say that the negation of

$$P: \text{ The integer } n \text{ is odd}$$

is

$$-P: \text{ The integer } n \text{ is not odd,}$$

but it is more useful to write

$$-P: \text{ The integer } n \text{ is even.}$$

(a) A: The triangle ABC is equilateral.
(b) B: For all real numbers $x \geq 0$, $x^2 - x \leq 0$.
(c) C: There exist integers m, n such that $m^2 = 2n^2$.
(d) D: If a and b are integers with $\gcd(a, b) = 2$, then there exist integers x and y such that $ax + by = 1$.

2. In this problem, let a, b, c be integers. Write down the converse of the following statements:
(a) If a and b satisfy $\gcd(a, b) = 1$, then there exist integers x, y such that $ax + by = 1$.
(b) If a divides bc, then either a divides b or a divides c.
(c) Show that (b) is false by providing a counterexample.

3. Suppose a, b, c denote the lengths of the three edges of some triangle in the plane. Write down first the converse and then the contrapositive of the following statement.

R: If the angle subtended by the sides of length a and b is 90 degrees, then $a^2 + b^2 = c^2$. Give your opinion on the validity of R, its converse and its contrapositive.

4. Consider the following sets

$$\begin{aligned} A &= \{x \in \mathbb{R} | x^2 - x \leq 0\} \\ B &= \{x \in \mathbb{R} | -(x-1)(x-3) \leq 0\} \\ C &= \{x \in \mathbb{R} | x \geq 1\}. \end{aligned}$$

(a) Determine $A \cap B$.
(b) Determine $A \cap C$.

5. Consider the function $f : (0, 1) \to (1, \infty)$ defined by $f(x) = 1/x$. Prove that f is bijective.

6. Consider functions $f : X \to Y$ and $g : Y \to Z$.
(a) Show that if $g \circ f : X \to Z$ is injective, then so is f. In fact, do this in at least 2 of the following 3 ways: i. direct proof, ii. proof by contradiction, iii. prove the contrapositive. (For each proof, indicate your method).
(b) Give an example of sets X, Y, Z and functions f, g as above such that $g \circ f$ is injective but g is not.

7. For the function $h(x) = \sin(e^{x-1})$, from $\mathbb{R}$ to $\mathbb{R}$, give two maps f and g from $\mathbb{R}$ to $\mathbb{R}$ such that $h = g \circ f$.

8. Let X be a set. Let S be the set of all subsets of X of cardinality one. In other words, Y is an element of S if and only if $Y \subseteq X$ and $|Y| = 1$.
(a) Prove or Disprove: X is equivalent to S.
(b) Prove or Disprove: S is a partition of X.

9. Consider the sets $A = \{0, 1\}$, $B = \{a, b, c\}$. List the elements of the sets $A \times A$, $A \times B$, $B \times A$, $A \times B \times A$.

10. (a) Suppose X, Y are **finite** sets of the same size, i.e. $|X| = |Y|$, and $f : X \to Y$ is an injective map. Show that f is also surjective, hence bijective.
(b) Suppose X, Y are **finite** sets of the same size i.e. $|X| = |Y|$, and $f : X \to Y$ is a surjective map. Show that f is also injective, hence bijective.
(c) Give an example of an injective map $f : X \to Y$ where X and Y are both infinite sets such that f is NOT surjective.
(d) Give an example of a surjective map $f : X \to Y$ where X and Y are both infinite such that f is not injective.

11. Suppose X, Y, Z are sets and $f : X \to Z$ and $g : Y \to Z$ are functions with target Z. Recall that $X \times Y$ is the set consisting of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. Using f and g, we define a special subset P of $X \times Y$ as follows:

$$P := \{(x, y) \in X \times Y | f(x) = g(y)\}.$$

The set P is called the *pullback* of f and g.

(a) There is an easy way to define a map $P \to Z$. What is it?

(b) Suppose $X = Y = Z = \mathbb{R}$, and suppose f and g are given by $f(x) = x^2$ for all $x \in X$ and $g(y) = y^2$ for all $y \in Y$, so that f and g are actually the same function. Note that P is now a subset of the set $X \times Y = \mathbb{R} \times \mathbb{R}$ which is just the xy-plane. Draw a picture of P (it is a pair of intersecting lines) and explain why your picture is correct.

(c) Same as (b) but now suppose $f(x) = e^x$ and $f(y) = e^y$. Show that P now consists of just one straight line.

12. Let $E = \{x \in \mathbb{Z} | x = 2m \text{ for some } m \in \mathbb{Z}\}$ be the set of even integers. Consider the "doubling" map $f : \mathbb{Z} \to E$ given by $f(m) = 2m$ for all $m \in \mathbb{Z}$.

(a) Show that f is injective.

(b) Show that f is surjective.

(c) You have shown that f gives an equivalence of sets (a one-to-one correspondence) between $\mathbb{Z}$ and E. Give a formula for the inverse function $f^{-1} : E \to \mathbb{Z}$. Do you believe that E and $\mathbb{Z}$ are equivalent sets? If not, why not? If not, how do you reconcile the non-equivalence of $\mathbb{Z}$ and E with (a) and (b)?

(d) Consider the map $g : O \to E$ from the set of odd integers O to the set of even integers E given by $g(k) = k - 1$. Verify that g is a well defined map from O to E.

(e) Give a formula for the inverse function $g^{-1} : E \to O$ of g.

(f) Now construct a bijective map from O to $\mathbb{Z}$ and give an explicit formula for it.

4. Extra Credit Problems

1. Using each of the ten digits 0,1,2,3,4,5,6,7,8,9 exactly once, it is possible to write numbers whose sum is 99. For instance,

$$4 + 5 + 6 + 7 + 30 + 28 + 19 = 99.$$

Prove or disprove: Using each of the ten digits 0,1,2,3,4,5,6,7,8,9 exactly once, it is possible to write numbers whose sum is 100.

2. Let S be the set of numbers N which can be expressed as the sum of a collection of numbers in which collection each of the digits 0—9 appears exactly once. For instance, in Problem A, we saw that $99 \in S$ and you were asked to determine whether $100 \in S$ or not.

(a) Determine, with proof, the smallest element of S.

(b) Show that S is finite by finding its largest element.

(c) Give a complete list of the elements of S, or a procedure for determining same.