UMASS AMHERST MATH 300 FALL '05, F. HAJIR

HOMEWORK 1: PROBLEM SOLVING, INDUCTIVE VS. DEDUCTIVE REASONING, AN INTRODUCTION TO PROOFS

1. Suppose you are given eight identical-looking and identical-feeling balls. You are also given a balance scale. (By placing balls on the two pans of the balance, you can establish whether the two collections weigh the same, and if not, which collection of balls is heavier than the other). You are given the information that of the 8 balls, 7 of them weigh exactly the same and one of them is slightly heavier than the others. You are allowed to use the balance twice, no more.

Proposition. There is a strategy guaranteed to reveal the identity of the heavy ball in just two weighings.

Determine the truth or falsity of this Proposition. If you believe it is true, describe a strategy and explain exactly why it is guaranteed to work. If you do not believe it to be true, then give a convincing argument for the lack of a strategy.

2. Olga, Laurel, Serge, and Zoltan have been hired as coaches for basketball, soccer, volleyball, and swimming. Zoltan, whose sister was hired to coach basketball, has never heard of Mia Hamm. Laurel doesn't like sports that involve balls. Do you have enough information to determine which sport each person was hired to coach? If yes, give the coach-sport correspondence. Be sure to give a detailed and careful explanation of the reasoning process by which you arrived at your conclusions.

Note. In case you are not sure how Mia Hamm fits into the picture: I should tell you that in this problem, you can assume that anyone who has never heard of Mia Hamm is not the soccer coach. Also, for those who do not know, Olga and Laural are women; Serge and Zoltan are not.

3. (a) Consider the pattern formed by the sums of the first n integers.

$$\begin{aligned} 1 &= 1 \\ 1+2 &= 3 \\ 1+2+3 &= 6 \\ 1+2+3+4 &= 10 \end{aligned}$$

Try to explain with a picture why the numbers $1, 3, 6, 10, 15, \ldots$ are called *triangular* numbers. [My own pet name for them is "bowling" numbers – can you explain that too?]

(b) The first bowling number is 1. Can you predict the tenth bowling number? Make a table of the first twelve bowling numbers. Can you determine hundredth bowling number? [When Gauss was six, he did this in a few seconds, astounding (and perhaps annoying?) his teacher.] Can you do it without writing them all down up to the hundreth? By messing around with a table of some bowling numbers, see if you can come up with a formula for the

nth bowling number. I'm not asking you to prove the formula is correct, but if you want to do that, give it a whirl!

(c) Twenty people come to a fancy dinner. Each guest shakes hands with every other guest exactly once. How many handshakes occur?

(d) Twenty points are marked on the perimeter of a circle. The line segment joining every pair of distinct marked points is drawn. How many line segments are drawn? (Try it as a thought experiment or as an actual activity).

(e) Now the number of people coming to the fancy dinner keeps fluctuating, so we just want to call the number of guests x and wish to have a formula $H(x)$, where $H(x)$ is the number of handshakes among x guests. What about where $H(x)$ is the number of line segments among x marked points on the circle?

4. Laura Beltis has 44 pennies and ten pockets in her vest. She wants to put all her pennies in her pockets in such a way that she has a different number of pennies in each pocket. Can she do it? If yes, tell Laura how, if not, explain to her (gently but cogently) why she cannot.

5. Consider a clock with line-thin hour-hand and minute-hand that move perfectly continuously. At noon and then again at midnight, the two hands line up perfectly.

(a) How many times between noon and midnight (counting midnight, but not noon) do the two hands line up again?

(b) Determine the **exact** times at which the two hands line up. Explain your reasoning very carefully.

Extra Credit Problems.

AN IMPORTANT NOTE ABOUT EXTRA CREDIT PROBLEMS. These problems are for your amusement and edification and to challenge or push you. Don't look here for an easy way to ameliorate your grade: a better way of doing that would be to concentrate more on other aspects of the course. There is no expectation here about whether you "should" be able to solve even a small part of any of these problems. In some cases, the solutions here are difficult and beyond this course entirely. In some cases, I do not myself know the solution to the problems stated here, and in fact the solution may be as yet unknown to *homo sapiens.* And in other cases, the solution may come to you easily, in a flash. The point here is to present what I think are interesting problems and see how far you can run with them, for your own entertainment and growth. You will receive a certain number of "bonus points" depending on the difficulty of the problem and how far you travelled into the solution. At the end of the semester, the student with the largest number of bonus points will receive a *fabulous* (sur)prize. I will be reluctant to reveal too many solutions too soon, as I want people time to hammer away at their favorites for quite some time.

A. Consider a generalization of Problem 1. Namely you are given n balls, 1 of which weighs slightly more than all the others, while the other $n-1$ are identical.

Let $s(n)$ be the smallest number $k \geq 0$ with the following property: if you are allowed to use the balance scale k times, then there is a strategy that will **always** allow you to identify the heavy ball.

For instance, in Problem 1, the Proposition states that $s(8) \leq 2$. Perhaps you will find it easy to convince yourself that $s(8) > 1$. Therefore, if the Proposition is true, then $s(8) = 2$.

Now, finally, here is the question: Can you find an explicit formula (or a procedure for determining) $s(n)$ as a function of n? Short of that, can you prove any bounds on this function? For example, is it true that $s(n) << n$? How about $s(n) \leq n/2$? How about $s(n) \leq n/3$?

Here are some strategies that may be useful (or they might just distract you from the true path, depending on your style of thinking!): make a chart of n versus $s(n)$ for small values of n. If you are not sure what $s(n)$ is exactly, at least record a guess or a bound. Turn the problem on its head: Say you allow yourself k uses of the balance and let $N(k)$ be the largest number of balls you can start with and still have a strategy guaranteed to reveal the one heavy ball via k weighings. If you can determine $N(k)$, you should be able to use that easily to determine $s(n)$.

B. Now let us generalize Problem 1 even a bit more. Let $1 \leq h \leq n$ be two whole numbers. Suppose you are given n balls, and the balls are of two types: h of the balls are slightly heavier (but all identical to each other) and the remaining $n-h$ are all identical to each other and slightly lighter than the other h. Now let $S(n,h)$ be the least number of uses of the balance scale that are needed in order to guarantee that you will be able to distinguish the heavy balls from the light. In particular, $S(n,1)$ is just the $s(n)$ of Problem A.

The ultimate question is the same as before: Determine $S(n,h)$ exactly if you can, but short of that, provide some bounds for it.

Start by determining $S(n,2)$ or bounds on it.

Have fun! Experiment, act it out, make charts, wild guesses, test your guesses, ask roommates for their crazy ideas,.... Can you see a relationship between $S(n,h)$ and $S(n,n-h)$? Is this useful at all? (Maybe it is and maybe it isn't).

C. Now generalize even more, why not?! Let $r \geq 1$ be an integer, and suppose $h_1, h_2, \cdots, h_r$ are r positive integers such that $h_1 + h_2 + \cdots + h_r = n$. Suppose you are given a collection of n balls h_1 of which have weight w_1, h_2 of which have weight w_2 etc. where $w_i \neq w_j$ for $i \neq j$. Let $S(h_1, \ldots, h_r)$ be the minimum number of weighings needed (as a function of $h_1, \ldots, h_r$) to guarantee one to be able to separate the balls into r bunches of equal-weight balls. What can you say about the function S?