Geometric Theory of Parshin Residues

Mikhail Mazin, University of Toronto

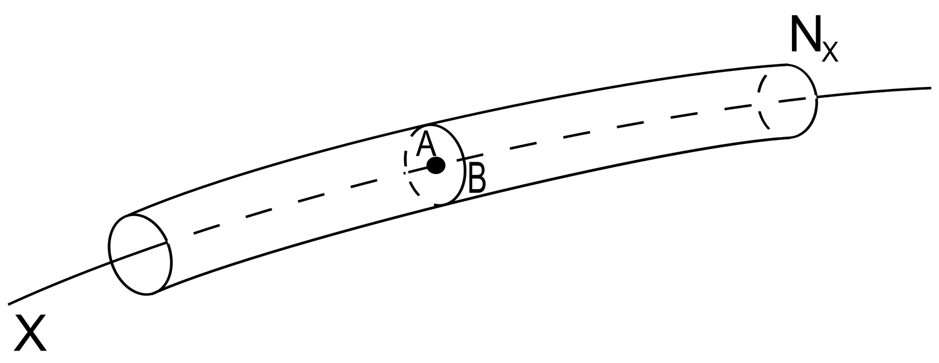
Part I

Coboundary homomorphisms for stratified spaces.

Let **S** be a stratification of the space V.

Let $X < Y$ be consecutive strata, let $\dim X = n$ and $\dim Y = k$. Then one can define the coboundary homomorphism $\phi_{X,Y} : H_*(X) \to H_{*+k-n-1}(Y)$.

Example 1.



Pic. 1

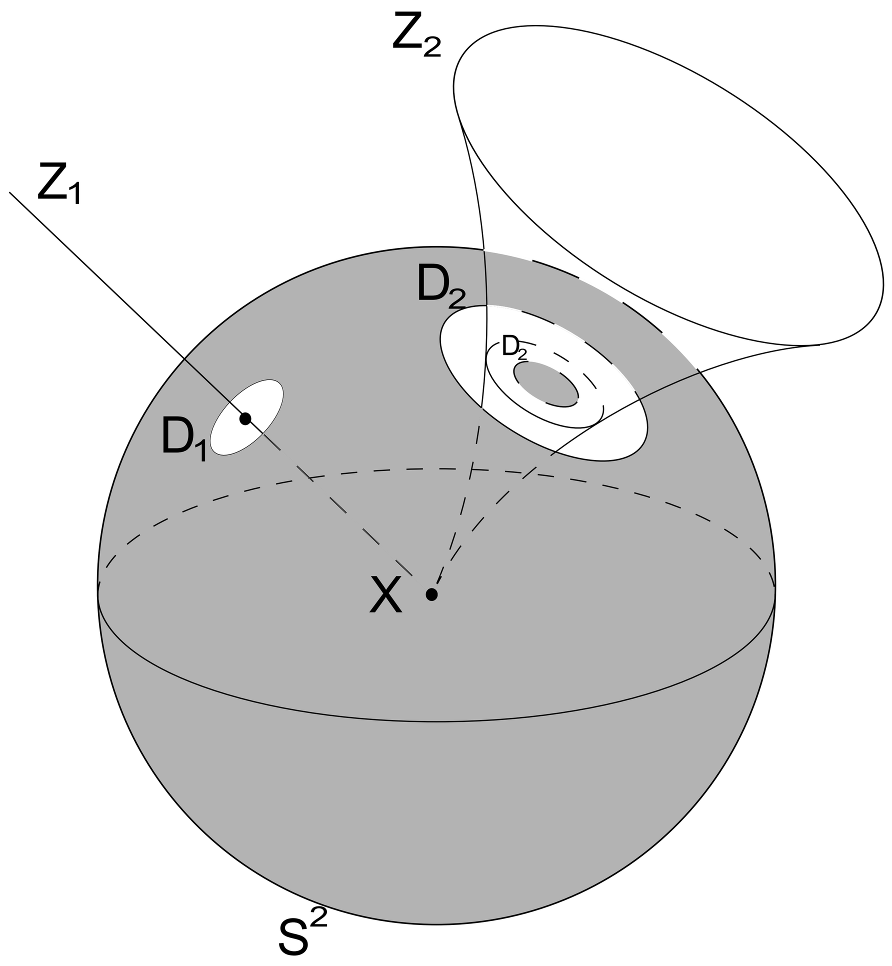
On the Pic. 1 we have two strata: X is a smooth curve in $\mathbb{R}^3$ and Y is the complement to X. Let A be a point on X. It represent the generator of $H_0(X)$. Let N_X be the boundary of a small neighborhood of X in $\mathbb{R}^3$. Then there is a smooth projection $\pi : N_X \to X$. Let $B = \pi^{-1}(A)$. Then B represents the class $\phi_{X,Y}([A]) \in H_1(Y)$.

Note, that regularity of the stratification implies existence of N_X in the general case, and the choice of consecutive strata $X < Y$ implies that $N_X \cap Y$ has compact fibers over X.

Theorem 1. *Let $X < Y$ be two strata. Let $Z_1, \dots, Z_n$ be all strata such that $X < Z_i < Y$. Suppose that $Z_1, \dots, Z_n$ are incomparable. Then*

$$\phi_{Z_1,Y} \circ \phi_{X,Z_1} + \phi_{Z_2,Y} \circ \phi_{X,Z_2} + \dots + \phi_{Z_k,Y} \circ \phi_{X,Z_k} = 0.$$

Example 2.



Pic. 2

On the Pic. 2 we have the following stratification of $\mathbb{R}^3 : X$ is the origin, Z_1 is an open half-line starting from the origin, Z_2 is a surface with an isolated singularity at the origin minus the origin, and $Y = \mathbb{R}^3 \backslash (X \cup Z_1 \cup Z_2)$. We take a small sphere S^2 with center at the origin. Then $\phi_{X,Z_i}([X]) \in H_{\dim Z_i - 1}(Z_i)$ is represented by the intersection $N_i = S^2 \cap Z_i$. Take a small neighborhood of N_i in S^2. Its boundary D_i represents the $\phi_{Z_i,Y} \circ \phi_{X,Z_i}([X]) \in H_1(Y)$. Then the sphere S^2 with the neighborhoods of N_i's deleted gives a two-dimensional chain in Y, which boundary is the union $D_1 \cup D_2$.

Application to the theory of Parshin residues.

Let $F = \{V_n \supset \dots \supset V_0\}$ be a flag of irreducible varieties. Let ω be a meromorphic n-form on V_n. Consider a stratification **S** of V_n, such that

1. $V_0, \dots V_{n-1}$ are unions of strata;
2. ω is regular on the top-dimensional stratum.

Definition 1. *We denote by $\breve{V}_k$ the k-dimensional stratum in V_k.*

Theorem 2. *Let $\Delta_F = \phi_{\breve{V}_{n-1},\breve{V}_n} \circ \dots \circ \phi_{\breve{V}_0,\breve{V}_1}([V_0]) \in H_n(\breve{V}_n)$. Then*

$$res_F(\omega) = \frac{1}{(2\pi i)^n} \int_{\Delta_F} \omega,$$

where $res_F(\omega)$ is the Parshin residue of ω at the flag F.

The next result gives a criteria for possible locations of non-trivial residues:

Theorem 3. *Fix a meromorphic n-form ω on V_n. Fix any stratification* **S** *of V_n, such that ω is regular in the top-dimensional stratum. Let $F = \{V_n \supset \dots \supset V_0\}$ be a flag of irreducible subvarieties, such that $res_F(\omega) \neq 0$. Then all elements of the flag F are unions of strata of the stratification* **S**.

The following Parshin's Reciprocity Law for residues now follows from Theorems 1, 2, and 3:

Theorem 4. *(Parshin, Beilinson, Lomadze) Let ω be a meromorphic n-form on V_n. Fix a partial flag of irreducible subvarieties $\{V_n \supset \dots \supset \hat{V}_k \supset \dots \supset V_0\}$, where V_k is omitted ($0 < k < n$). Then*

$$\sum_{V_{k+1} \supset X \supset V_{k-1}} res_{V_n \supset \dots \supset X \supset \dots \supset X_0}(\omega) = 0,$$

where the sum is taken over all irreducible k-dimensional subvarieties X, such that $V_{k-1} \supset X \supset V_{k+1}$. (In this formula only finitely many summands are non zero.)

Part II.

Proper preimage of a hypersurface.

Definition 2. *Let $f : X \to Y$ be a branched covering. Let $H \in Y$ be a hypersurface. Then the proper preimage $H_f \subset X$ of H is the union of those irreducible components H_f^i of the full preimage $f^{-1}(H)$ for which $f(H_f^i)$ has codimension 1 in Y.*

Lemma 1. *$f|_{H_f} : H_f \to H$ is a branched covering.*

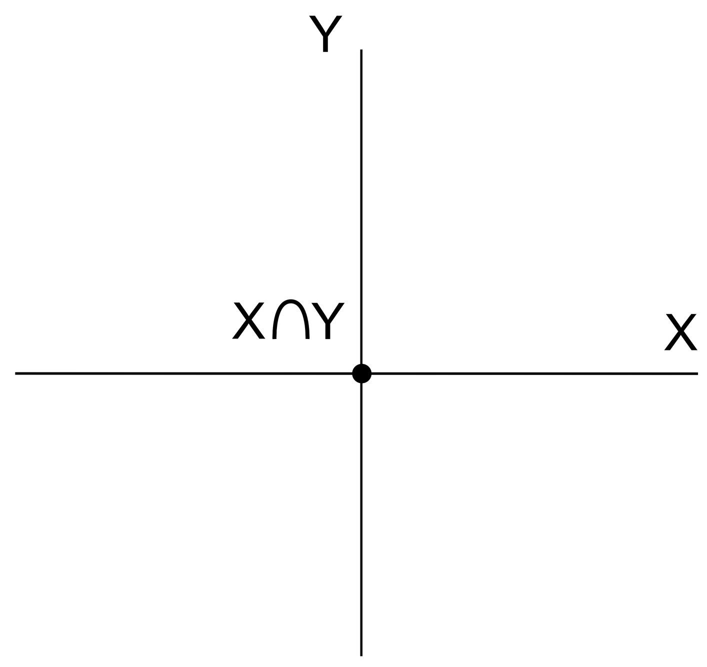
Resolution of Singularities for Flags.

Theorem 5. *Let V be an algebraic variety. Let $Y_1, \dots, Y_K$ be closed subvarieties in X. Then there exists a resolution of singularities $\pi : \overline{X} \to X$ such that:*

1. For any $m = 1, \dots, K$ the preimage $\pi^{-1}(Y_m)$ is a union of exceptional hypersurfaces;
2. Let H_i, H_j be exceptional hypersurfaces and $H_i \cap H_j \neq \emptyset$. Then $\pi(H_i) \subset \pi(H_j)$ or $\pi(H_i) \supset \pi(H_j)$;
3. Let $H_{i_1}, \dots, H_{i_k}$ be exceptional hypersurfaces, $C := H_{i_1} \cap \dots \cap H_{i_k} \neq \emptyset$, and $\pi(H_{i_1}) \subset \dots \subset \pi(H_{i_k})$. Then C is irreducible and $\pi(C) = \pi(H_{i_1})$.

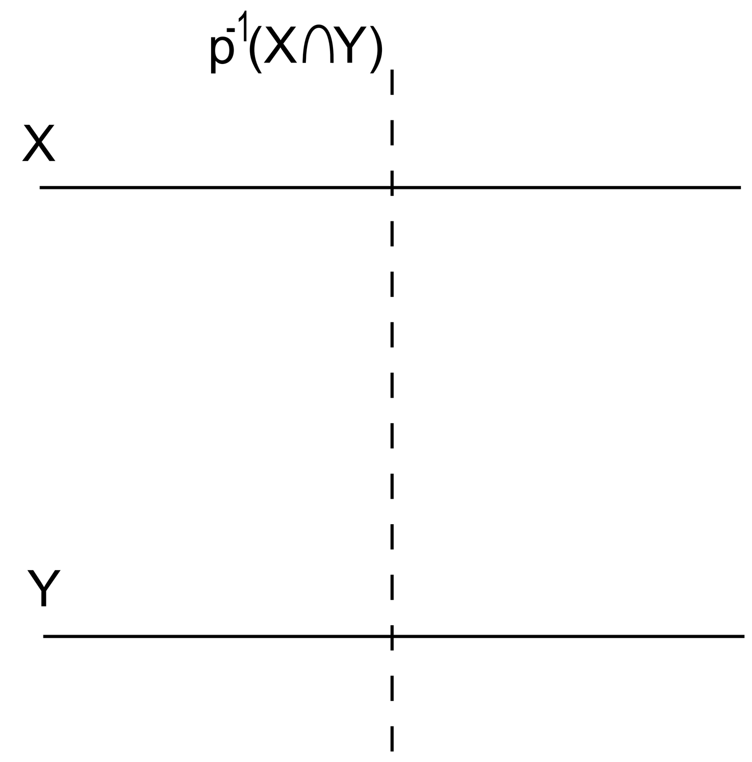
The existence of a resolution satisfying condition 1 follows from the classical Hironaka's Theorem. In order to obtain conditions 2 and 3 one needs to do some additional blow-ups with centers in the intersections of exceptional hypersurfaces.

Example 3.



Pic. 3

Suppose that we have a smooth space V and two smooth hypersurfaces X and Y intersecting transversally. From the first sight it looks like there is nothing to resolve: one just needs to say, that X and Y are exceptional hypersurfaces. However, the condition 2 is not satisfied. To obtain conditions 2 and 3 one needs to blow-up the intersection $X \cap Y$:



Pic. 4

Easy to see that now conditions 2 and 3 are satisfied.

Parshin residues via resolution of singularities.

Let $F = \{V_n \supset \dots \supset V_0\}$ be a flag of irreducible subvarieties and ω be a meromorphic n-form on V_n. We apply Theorem 5 to V_n with subvarieties $V_{n-1}, \dots, V_0$, and the divisor of poles of ω.

Let $\pi : \overline{V}_n \to V_n$ be the resolution and $\mathcal{D} = \{H_1, \dots, H_N\}$ be the set of exceptional hypersurfaces.

Let $\mathcal{D}_k = \{H_i \in \mathcal{D} : \pi(H_i) = V_k\}$ and $D_k = \bigcup_{H_i \in \mathcal{D}_k} H_i$. Let also $\overline{V}_{n-1} \supset \dots \supset \overline{V}_0$ be the flag of consecutive proper preimages of the flag $V_n \supset \dots \supset V_0$ (i.e. $\overline{V}_k$ is the proper preimage of V_k under $\pi|_{\overline{V}_{k+1}}$).

Lemma 2. *$\overline{V}_k = D_{n-1} \cap \dots \cap D_k$ and $\overline{V}_k$ is smooth for all $k = 0, \dots, n$.*

$\overline{V}_0$ is a finite set of point. At each point of $\overline{V}_0$ exactly n exceptional hypersurfaces meet.

Definition 3. *Let $a \in \overline{V}_0$ and $(x_1, \dots, x_n)$ be local coordinates near a, such that the exceptional hypersurfaces coincide with the coordinate hyperplanes in a neighborhood of a. Denote*

$$\gamma_a = \{|x_1| = |x_2| = \dots = |x_n| = \epsilon\}.$$

Theorem 6.

$$res_F(\omega) = \frac{1}{(2\pi i)^n} \sum_{a \in \overline{V}_0} \int_{\gamma_a} \pi^*(\omega)$$

We also use the resolution of singularities to study the local geometry near the flag $V_n \supset \dots \supset V_0$. In particular, we study the area of convergence of the iterated Laurent power series, involved in the original Parshin's definitions.