Homological Projective Duality for Gr(3,6)

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Introduction

• Homological Projective Duality (HPD) is a homological extension of the classical notion of projective duality and came as an attempt to answer the question of whether having a semiorthogonal decomposition on $D^b(X)$ allows one to construct a decomposition of $D^b(X_H)$, where X_H is a hyperplane section of X. In general the answer is no, however in the context of HPD much can be said about this.

• One starts with a smooth (noncommutative) algebraic variety X with a map $X \to \mathbb{P}(V)$ and associates to it a smooth (noncommutative) algebraic variety Y with a map $Y \to \mathbb{P}(V^*)$ into the dual projective space (the classical projective dual variety of X will be given by the critical values of this second map). This construction will depend on a specific kind of a semiorthogonal decomposition of $D^b(X)$ called a Lefschetz decomposition.

Definitions

Let X be an algebraic variety with $\mathcal{O}_X(1)$ an ample line bundle on it.

Definition. *A* ***Lefschetz decomposition*** *of the derived category $D^b(X)$ is a semiorthogonal decomposition of the form $D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), ..., \mathcal{A}_{k-1}(k-1)\rangle$, where $0 \subset \mathcal{A}_{k-1} \subset \mathcal{A}_{k-2} \subset ... \subset \mathcal{A}_0 \subset D^b(X)$ is a chain of admissible subcategories of $D^b(X)$ and (i) means twisting by $\mathcal{O}(i)$.*

Definition. *An algebraic variety Y with a projective morphism $g : Y \to \mathbb{P}(V^*)$ is called* ***Homologically Projective Dual*** *to $f : X \to \mathbb{P}(V)$ with respect to a given Lefschetz decomposition as above, if there exists an object $\mathcal{E} \in D^b(\mathcal{X} \times_{\mathbb{P}(V^*)} Y)$ (where $\mathcal{X} \subset X \times \mathbb{P}(V^*)$ is the universal hyperplane section of X) such that the kernel functor $\Phi = \Phi_{\mathcal{E}} : D^b(Y) \to D^b(\mathcal{X})$ is fully faithful and gives the following semiorthogonal decomposition*

$$D^b(\mathcal{X}) = \langle \Phi(D^b(Y)), \mathcal{A}_1(1) \boxtimes D^b(\mathbb{P}(V^*)), ..., \mathcal{A}_{k-1}(k-1) \boxtimes D^b(\mathbb{P}(V^*))\rangle.$$

Homological Projective Duality

Given a linear subspace $L \subset V^*$, where $dim(V) = N$, we consider the linear sections of X and Y: $X_L = X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L = X \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$, where $L^\perp \subset V$ is the orthogonal subspace to $L \subset V^*$.

Theorem (A.Kuznetsov)**.**
If Y is Homologically Projective Dual to X then

(i) Y is smooth and $D^b(Y)$ admits a dual Lefschetz decomposition $D^b(Y) = \langle \mathcal{B}_{j-1}(1-j), ..., \mathcal{B}_1(-1), \mathcal{B}_0\rangle$, where $0 \subset \mathcal{B}_{j-1} \subset \mathcal{B}_{j-2} \subset ... \subset \mathcal{B}_0 \subset D^b(Y)$.

(ii) For any linear subspace $L \subset V^*$, $dim(L) = r$, such that $dimX_L = dimX - r$ and $dimY_L = dimY + r - N$, there exists a triangulated category $\mathcal{C}_L$ and semiorthogonal decompositions $D^b(X_L) = \langle \mathcal{C}_L, \mathcal{A}_r(1), ..., \mathcal{A}_{i-1}(i-r)\rangle$ and $D^b(Y_L) = \langle \mathcal{B}_{j-1}(N-r-j), ..., \mathcal{B}_{N-r}(-1), \mathcal{C}_L\rangle$.

Gr(3,6)

Let W be a vector space, dim $W = 6$ and let $X = Gr(3, W)$ be the Grassmannian of planes in W. Let $\mathcal{U}$ be the tautological rank 3 vector bundle on $Gr(3, W)$.

We will now define a Lefschetz collection for X, then describe the main constructions necessary to define the HPD of X and finally state the main result.

Lefschetz collection

The bundles $E_0 = \mathcal{O}_X$, $E_1 = \mathcal{U}^*$, $E_2 = \Lambda^2\mathcal{U}^*$, $E_3 = \Sigma^{2,1}\mathcal{U}^*$ are exceptional on X. Define $\mathcal{A}_0 = \mathcal{A}_1 = \langle E_0, E_1, E_2, E_3\rangle$ and $\mathcal{A}_i = \langle E_0, E_1, E_2\rangle$, for $i = 2, \ldots, 5$.

Proposition. *On X, the subcategories $\mathcal{A}_5 \subset \ldots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ give a Lefschetz decomposition $D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_5(5)\rangle$.*

This is proven using an induction step (considering $0 \neq s \in H^0(X, \mathcal{U}^*)$ and taking its zero locus $Z_s \cong Gr(3, 5) \subset X$) and some manipulations with exact sequences.

Main Constructions

• Consider the universal discriminant $Z \subset \mathbb{P}(W) \times \mathbb{P}(\Lambda^3 W^*)$, where Z consists of pairs (w, λ) such that $Gr(2, 5)_w \cap H_\lambda$ is singular ($Gr(2, 5)_w \subset Gr(3, 6) \subset \mathbb{P}(\Lambda^3(W))$ is the fiber of $\mathbb{P}_X(\mathcal{U}) \to \mathbb{P}(W)$ and $H_\lambda \subset \mathbb{P}(\Lambda^3(W))$ is the hyperplane corresponding to λ). Actually $Z = \{(w, \lambda) \mid rank(\lambda \lrcorner w) \leq 2\}$.

• Geometric considerations that we will mention later imply that the HPD of X will be a category $\mathcal{C}$ such that $D^b(Z)$ will be generated by 3 copies of $\mathcal{C}$, which roughly means that we can find a structure $Z \to M$ of a $\mathbb{P}^2$-bundle on Z.

• Use that $GL(W)$ acts on $\Lambda^3(W)$ with an open orbit. A generic 3-form can be written as $x_0 \wedge x_1 \wedge x_2 + x_3 \wedge x_4 \wedge x_5$, so the generic fibers of $Z \to \mathbb{P}(\Lambda^3 W^*)$ are $Z_\lambda = \{x_0 = x_1 = x_2 = 0\} \sqcup \{x_3 = x_4 = x_5 = 0\} = \mathbb{P}^2 \sqcup \mathbb{P}^2$. Stein factorization then gives $Z \to M \to \mathbb{P}(\Lambda^3 W^*)$, where the first map is generically a $\mathbb{P}^2$-bundle and the second one is a double cover.

• The generic form of λ shows that the map
$Y = \mathbb{P}_{Gr(3,W^*)\times Gr(3,W^*)}(\mathcal{O}(-1, 0) \bigoplus (\mathcal{O}(0, -1))) \dashrightarrow \mathbb{P}(\Lambda^3 W^*)$ is a rational map on the 2-fold covering M, which after blowing up the diagonally embedded $Gr(3, W^*)$ gives a regular map $p : \widetilde{Y} \to M$.

• Look at $Y \to Gr(3, W^*) \times Gr(3, W^*)$ and consider $\mathcal{A}$ and $\mathcal{B}$ to be the tautological bundles on the respective Grassmannians. We denote their pullbacks to Y (and $\widetilde{Y}$) with the same letters and we use $\mathcal{O}(-1)$ for the pullback of $\mathcal{O}(-1)$ from $\mathbb{P}(\Lambda^3 W^*)$ to $\widetilde{Y}$.

• Consider now the following bundles on $\widetilde{Y}$:
F_0 is the nontrivial extension $0 \to \mathcal{O}(-1) \to F_0 \to \mathcal{A} \boxtimes \Lambda^2(\mathcal{B}) \to 0$, $F_1 = \Lambda^2(\mathcal{B})$, $F_2 = \mathcal{A}$, $F_3 = \mathcal{O}_{\widetilde{Y}}$. Let $F = F_0 \bigoplus F_1 \bigoplus F_2 \bigoplus F_3$ and define $\mathcal{R} = p_* EndF$.

• Let $\mathcal{B}_i = \langle F_0, F_1, F_2, F_3\rangle$, for $0 \leq i \leq 13$ and $\mathcal{B}_j = \langle F_3\rangle$, for $14 \leq j \leq 17$. Define $\mathcal{C} = \langle \mathcal{B}_{17}(-17), ..., \mathcal{B}_1(-1), \mathcal{B}_0\rangle$.

Theorem. *The noncommutative resolution of singularities $(M, \mathcal{R})$ of M is Homologically Projective Dual to the Grassmannian $X = Gr(3, W)$ with respect to the Lefschetz decomposition constructed above. The corresponding Lefschetz decomposition of $D^b(M, \mathcal{R})$ is given by the $\mathcal{B}_i$'s above.*

How does the HPD come up

• For a smooth projective variety $X \subset \mathbb{P}(V)$ with a Lefschetz decomposition of $D^b(X)$ corresponding to $\mathcal{O}_X(1)$, and for any hyperplane section X_H of X, we have that $D^b(X_H) = \langle \mathcal{C}_H, \mathcal{A}_1(1), ..., \mathcal{A}_{k-1}(k-1)\rangle$ (the composition $\mathcal{A}_i(i) \to D^b(X) \to D^b(X_H)$ is fully faithful). We consider the family $\{\mathcal{C}_H\}_{H \in \mathbb{P}(V^*)}$. Finding the homological projective dual Y as above means that this family is "geometric", i.e. for $Y \to \mathbb{P}(V^*)$ and for all H we have that $\mathcal{C}_H \cong D^b(Y_H)$, where Y_H is the fiber over $H \in \mathbb{P}(V^*)$.

• The way to do this is to actually look at the universal variant of this. If we let $\mathcal{X} \subset X \times \mathbb{P}(V^*)$ be the universal hyperplane section of X, we have a decomposition $D^b(\mathcal{X}) = \langle \mathcal{C}, \mathcal{A}_1(1) \boxtimes D^b(\mathbb{P}(V^*)), ..., \mathcal{A}_{k-1}(k-1) \boxtimes D^b(\mathbb{P}(V^*))\rangle$ and check that $\mathcal{C}$ is equivalent to $D^b(Y)$.

• Now for $Gr(3, 6)$, look at $D^b(\mathbb{P}_{X_H}(U))$. Notice that the fibers of $\mathbb{P}_{X_H}(U) \to \mathbb{P}(W)$ are hyperplane sections of $Gr(2, 5)$ and that $\mathbb{P}_{X_H}(U) \to X_H$ is a $\mathbb{P}^2$-bundle. Since the derived category of a hyperplane section of $Gr(2, 5)$ is known to have some nontrivial part only if it is singular, we get that $D^b(Z_H)$ (where Z_H is the discriminant locus of the first map) should contain 3 copies of $\mathcal{C}_H$.

• Thus $D^b(Z)$ should have 3 copies of $\mathcal{C}$ and since $Z \to M$ is generically a $\mathbb{P}^2$-bundle, we see that $D^b(M)$ appropriately resolved should contain one copy of $\mathcal{C}$.

Proof ingredients

• Show that on $X \times \widetilde{Y}$ there is a complex of vector bundles

$$\{E_0 \boxtimes F_0 \to E_1 \boxtimes F_1 \oplus E_2 \boxtimes F_2 \to E_3 \boxtimes F_3\} \cong i_*\mathcal{E},$$

that is quasiisomorphic to a coherent sheaf $\mathcal{E}$ supported on the incidence variety $I(X, \widetilde{Y}) \overset{i}{\hookrightarrow} X \times \widetilde{Y}$ (where $I(X, \widetilde{Y}) \cong \mathcal{X} \times_{\mathbb{P}^*} \widetilde{Y}$ is the preimage of the incidence variety on $\mathbb{P} \times \mathbb{P}^*$).

• Use that $\mathcal{X}$ is a divisor inside $X \times \mathbb{P}^*$ (and j is the embedding) and thus we have a distinguished triangle of functors $D^b(\mathcal{X}) \to D^b(\mathcal{X})$

$$j^* j_* \to id \to \mathcal{O}_{\mathcal{X}}(-1, -1)[2]$$

• Conclude that the functor $\Phi_{i_*\mathcal{E}} : D^b(\widetilde{Y}) \to D^b(\mathcal{X})$ gives a fully faithful embedding of $\mathcal{C}$ into $D^b(\mathcal{X})$.

• Note that $\Phi^*_{i_*\mathcal{E}} \circ \pi^* : D^b(X) \to D^b(\widetilde{Y})$ (where $\pi : \mathcal{X} \to X$) is fully faithful on the subcategory $\mathcal{A}_0 \subset D^b(X)$ and that its image is $\mathcal{B}_0 \subset D^b(\widetilde{Y})$ and use a theorem of Kuznetsov to conclude that $D^b(M, \mathcal{R})$ is the HPD of X.

Future

• Apply the main theorem to linear sections.
• Do a similar construction for Gr(3,7).