Math 797AS Homework 4

Paul Hacking

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(1) (a) Let $X = (f = 0) \subset \mathbb{C}^{n+1}$ be a smooth hypersurface. The tangent space T_pX to X at a point p is identified with the affine hyperplane

$$\left(\sum \frac{\partial f}{\partial z_i}(p)(z_i - z_i(p)) = 0\right) \subset \mathbb{C}^{n+1}.$$

Show that if H is a hyperplane in $\mathbb{C}^{n+1}$ through $p \in X$, then the divisor $H|_X$ has multiplicity ≥ 2 at p iff $H = T_pX$.

[Recall if X and Y are complex manifolds, $f \colon X \to Y$ is a holomorphic map, and D is a divisor on Y such that $f(X)$ is not contained in the support of D, then we define f^*D as follows: let $U \subset Y$ be an open set such that $D|_U$ is the principal divisor (g) associated to a meromorphic function g on U, then $f^*D|_{f^{-1}U} := (g \circ f)$. If f is a closed embedding we also write $D|_X$ for f^*D. If D is an effective divisor on a complex manifold X and $p \in X$ is a point, we define the *multiplicity* $\operatorname{mult}_p D$ of D at p as follows: in a small neighborhood of p the divisor D is the principal divisor (f) associated to a holomorphic function f; expand f as a power series $\sum a_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}$ in local coordinates $z_1, \ldots, z_n$ at p, then $\operatorname{mult}_p D := \min\{\sum i_j \mid a_{i_1,\ldots,i_n} \neq 0\}$.]

(b) Suppose $X \subset \mathbb{P}^3_{(Z_0:Z_1:Z_2:Z_3)}$ is a smooth hypersurface of degree d which contains the line $L = (Z_0 = Z_1 = 0) \subset \mathbb{P}^3$. Show that the rational map $\varphi \colon X \dashrightarrow \mathbb{P}^1$ defined by $(Z_0 : Z_1)$ is a morphism.

[Hint: The rational map φ is defined by the linear system δ of hyperplane sections $H|_X$ of X containing L. Each element $D = H|_X \in \delta$ can be written as $L + M$ for some divisor M on X with support contained in H. So the fixed divisor of the linear system

δ is the line L, and (removing the fixed divisor) the linear system δ' given by the curves M defines the same rational map. Now use part (a) to show that the linear system δ' has no basepoints (that is, for all $p \in X$ there exists $M \in \delta'$ such that $p \notin M$) so that φ is a morphism.]

(c) Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree 4 and suppose that X contains a line L. Show that there is a morphism $X \to \mathbb{P}^1$ such that the general fiber is a smooth curve of genus 1.

[Note: The general fiber is smooth by Sard's theorem.]

(2) Let $X = \mathbb{C}^2$ and $\pi \colon \tilde{X} \to X$ be the blowup of the point $p = (0,0) \in X$. Let $p \in C \subset X$ be a curve. Recall that the strict transform $C' \subset \tilde{X}$ of C is defined by $C' = \overline{\pi^{-1}(C \setminus \{p\})}$. For each of the following curves C, use the explicit description of the blowup π in charts to compute the strict transform C' and verify that C' is smooth.

(a) $C = (z_2^2 = z_1^2(z_1 + 1))$.

(b) $C = (z_2^2 = z_1^3)$.

(3) Let $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona transformation $(Z_0 : Z_1 : Z_2) \mapsto (Z_1 Z_2 : Z_0 Z_2 : Z_0 Z_1)$.

(a) Show that $\varphi^2 = \mathrm{id}$. In particular, φ is a birational map.

(b) Compute the base locus of the linear system δ defining φ.

(c) Let $\pi \colon X \to \mathbb{P}^2$ denote the composition of the blow ups of the base points of δ. Show that there is a morphism $\tilde{\varphi} \colon X \to \mathbb{P}^2$ such that $\tilde{\varphi} = \varphi \circ \pi$.

(d) Show that $\tilde{\varphi}$ contracts the strict transforms of the coordinate lines $(Z_0 = 0), (Z_1 = 0), (Z_2 = 0)$ to the points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and has no other exceptional curves.

(e) Show that the exceptional curves of $\tilde{\varphi}$ are (-1)-curves. (Recall that we say a curve E on a smooth surface S is a (-1)-*curve* if $E \simeq \mathbb{P}^1$ and $E^2 = -1$.) So, by the Castelnuovo contractibility criterion, $\tilde{\varphi}$ is a composition of blowups.

(4) Let $\varphi \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ be the rational map $((X_0 \colon X_1), (Y_0 \colon Y_1)) \mapsto (X_0 Y_0 : X_1 Y_0 : X_0 Y_1)$.

(a) Show that φ is a birational map by finding an explicit formula for its inverse ψ (use the Segre embedding

$$\iota\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \quad ((X_0 : X_1), (Y_0 : Y_1)) \mapsto (X_0Y_0 : X_1Y_0 : X_0Y_1 : X_1Y_1)$$

and express ψ as a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ which factors through $\iota(\mathbb{P}^1 \times \mathbb{P}^1)$).

(b) Compute the base locus of the linear system δ defining φ.

(c) Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blowup of the base points of φ. Show that there is a morphism $\tilde{\varphi}\colon X \to \mathbb{P}^2$ such that $\tilde{\varphi} = \varphi \circ \pi$.

(d) Show that $\tilde{\varphi}$ contracts the strict transforms of the curves $(X_0 = 0), (Y_0 = 0) \subset \mathbb{P}^1 \times \mathbb{P}^1$ to the points $(0 : 1 : 0), (0 : 0 : 1)$ and has no other exceptional curves.

(e) Show that the exceptional curves of $\tilde{\varphi}$ are (-1)-curves.

(5) (a) Let X be a smooth projective surface and $\pi\colon \tilde{X} \to X$ the blowup of a point on X. Show that $K_{\tilde{X}}^2 = K_X^2 - 1$.

(b) Now let $X \subset \mathbb{P}^3$ be a smooth cubic surface.

i. Show that $-K_X$ is very ample, and compute $(-K_X)^2$. [Hint: Use the adjunction formula for $X \subset \mathbb{P}^3$.]

ii. Recall that in class we showed that there is a birational morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1$ (assuming the existence of two skew lines $L_1, L_2 \subset X$; see [UAG], Chapter 7 for the proof of this fact). Moreover, we showed that any birational morphism $f\colon X \to Y$ of smooth projective surfaces is a composition of blowups. Using part (a) deduce that X is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in 5 points or, equivalently (by Q4), the blowup of $\mathbb{P}^2$ in 6 points.

iii. Now suppose Y is the blowup of six points in $\mathbb{P}^2$. Show that if $-K_Y$ is ample then no two of the points coincide, no three are collinear, and there does not exist a conic passing through all six points. (Conversely, if these conditions are satisfied then $-K_Y$ is very ample and the linear system $|-K_Y|$ defines an embedding of Y in $\mathbb{P}^3$ with image a cubic surface. See e.g [GH78], p. 480–483.) [Hint: Show that if one of the conditions fails then Y contains a curve C such that $C \simeq \mathbb{P}^1$ and $C^2 \leq -2$. Then $-K_Y \cdot C \leq 0$ (why?) so $-K_Y$ is not ample (why?).]

(6) Let $\varphi\colon X \dashrightarrow Y \subset \mathbb{P}^m$ be a rational map from a smooth projective surface X to a projective variety Y defined by a linear system $\delta \subset |D|$ without fixed divisors. We showed in class that there is a composition of blowups $\pi\colon \tilde{X} \to X$ and a morphism $\tilde{\varphi}\colon \tilde{X} \to Y$ such that $\tilde{\varphi} = \varphi \circ \pi$. Prove that at most D^2 blowups are required.

(7) Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. Suppose that X contains a line $L \subset \mathbb{P}^3$. Show that, regarding L as a curve on X, its self-intersection number $L \cdot L$ is given by $L^2 = -(d-2)$.

[Hint: Let $H \subset \mathbb{P}^3$ be a general hyperplane containing L and consider $H|_X = L + Y$. Alternatively, use the adjunction formula.]

(8) Let $a, b \in \mathbb{N}$. Let $p \in C \subset X$ be a germ of a curve on a smooth surface such that for some choice of local coordinates at $p \in X$ the curve C has local equation $z_1^a = z_2^b$. Compute the normalization $\nu\colon \tilde{C} \to C$ of C.

[Hint: Use the analytic construction of the normalization described in class, cf. [GH78], p. 498-500. Note that the germ $(p \in C)$ is irreducible iff $\gcd(a, b) = 1$ (why?).]

(9) Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree 5 with a unique singularity $p \in C$. Suppose that for some choice of local analytic coordinates z_1, z_2 at $p \in \mathbb{P}^2$ the curve C has local equation $z_1^2 = z_2^n$. Show that $n \leq 13$.

[Remark: In fact this bound is sharp by [W96], see p. 268, case G4.]

References

[GH78] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley, 1978.

[UAG] M. Reid, Undergraduate algebraic geometry, C.U.P., 1988; available on the author's website at `https://homepages.warwick.ac.uk/staff/Miles.Reid/MA4A5/UAG.pdf` .

[W96] C. Wall, Highly singular quintic curves, Math. Proc. Cambridge Philos. Soc. 119 (1996), no. 2, 257–277.