Math 797AS Homework 3

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(1) Let X be a compact complex curve (or Riemann surface) and $L \to X$ a holomorphic line bundle. We define the *degree* of L as follows: Let s be a generic C^∞ section of L. For each zero $p \in X$ of s, we assign a sign as follows: let $z = x + iy$ be a local coordinate at p and choose a local trivialization of $\phi\colon L|_U \to U \times \mathbb{C}$ in a neighbourhood U of p, so that $\phi(s(q)) = (q, f(q))$ for $q \in U$, where $f = u + iv\colon U \to \mathbb{C}$ is a C^∞ function. Then the sign of p is given by the sign of the determinant of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

The degree of L is then defined to be the signed count of the zeroes of s. (Equivalently, $\deg L$ is the intersection number of $s(X)$ and the zero section of L, where both sections are given the orientations induced by the orientation of X.) In terms of the first Chern class, $\deg L \in \mathbb{Z}$ is equal to $c_1(L) \in H^2(X, \mathbb{Z})$ where we identify $H^2(X, \mathbb{Z}) = H_0(X, \mathbb{Z}) = \mathbb{Z}$ using the orientation of X and Poincaré duality.

(a) If s is a meromorphic section of L and $(s) = \sum n_i p_i$ is the divisor of zeroes and poles of s, show that $\deg L$ equals the degree $\sum n_i$ of (s), that is, the number of zeroes minus the number of poles of s, counting multiplicities.

[Hints: (1) If L has a meromorphic section s with $(s) = D$, then $\mathcal{L} \simeq \mathcal{O}_X(D)$. (2) The maps $\operatorname{Cl}(X) \to \operatorname{Pic} X$, $D \mapsto \mathcal{O}_X(D)$ and $\deg\colon \operatorname{Pic}(X) \to \mathbb{Z}$ are group homomorphisms. (3) The case $(s) = p$ follows from the definition of degree.]

(b) Deduce that if $\deg L < 0$ then $H^0(X, L) = 0$

(2) Let X be a compact complex curve of genus g. Let $L \to X$ be a holomorphic line bundle. Assume that L admits a meromorphic section s, so that $\mathcal{L} \simeq \mathcal{O}_X(D)$ where $D = (s)$. (Remark: Existence of a meromorphic section holds because X is projective by the Kodaira embedding theorem, cf. [GH78], p. 213-4.)

(a) Use the exact sequence of sheaves

$$0 \to \mathcal{O}_X(D-p) \to \mathcal{O}_X(D) \to \mathbb{C}(p) \to 0$$

(cf. HW2Q3b) and induction to prove the Riemann–Roch formula in the form

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \deg L$$

.

(b) Use the Dolbeault theorem $H^q(\Omega_X^p) \simeq H^{p,q}$ to prove $\chi(\mathcal{O}_X) = 1 - g$.

(3) Let X be a compact complex curve of genus g. Let $\omega_X = \Omega_X$ be the canonical line bundle. Show that $\chi(\omega_X) = g - 1$ and deduce the Hopf index theorem

$$\deg \omega_X = -e(X) = 2g - 2.$$

[Hint: Use the Dolbeault theorem and Riemann–Roch]

(4) Let X be a compact complex manifold and $L \to X$ a holomorphic line bundle. Let $s_0, \ldots, s_m$ be a basis of $H^0(X, \mathcal{L})$, and

$$Z = (s_0 = \cdots = s_m = 0) \subset X$$

the *base locus* of L. Let

$$\varphi \colon X \setminus Z \to \mathbb{P}^m, \quad p \mapsto (s_0(p) : s_1(p) : \cdots : s_m(p))$$

be the associated holomorphic map of complex manifolds.

(a) For $p \in X$, let $\mathcal{I}_p \otimes \mathcal{L}$ be the sheaf of sections of L vanishing at p. Then we have an exact sequence of sheaves on X

$$0 \to \mathcal{I}_p \otimes \mathcal{L} \to \mathcal{L} \to L_p \to 0$$

where L_p denotes the skyscraper sheaf at p with stalk the fiber L_p of L over p. Show that L is *basepoint free*, that is, $Z = \emptyset$, iff $H^0(X, \mathcal{L}) \to L_p$ is surjective for all $p \in X$.

(b) Assume L is basepoint free. For $p, q \in X$, $p \neq q$, let $\mathcal{I}_{p,q} \otimes \mathcal{L}$ be the sheaf of sections of L vanishing at p and q. We have the exact sequence of sheaves

$$0 \to \mathcal{I}_{p,q} \otimes \mathcal{L} \to \mathcal{L} \to L_p \oplus L_q \to 0.$$

Show that $\varphi\colon X \to \mathbb{P}^m$ is injective iff $H^0(X, L) \to L_p \oplus L_q$ is surjective for all $p, q \in X$, $p \neq q$.

(c) Assume L is basepoint free and φ is injective. For $p \in X$, let $\mathcal{I}_p^2 \otimes \mathcal{L}$ be the sheaf of sections of $\mathcal{L}$ vanishing to order 2 at p (that is, in a local trivialization of L, the section is given by a holomorphic function f such that $f(p) = 0$ and $f'(p) = 0$). Then we have an exact sequence of sheaves on X

$$0 \to \mathcal{I}_p^2 \otimes \mathcal{L} \to \mathcal{I}_p \otimes \mathcal{L} \to T^*_{X,p} \otimes L_p \to 0$$

where $T^*_{X,p}$ denotes the dual of the tangent space to X at p. Show that φ is a closed embedding (isomorphism onto a closed submanifold) iff $H^0(\mathcal{I}_p \otimes \mathcal{L}) \to T^*_{X,p} \otimes L_p$ is surjective for all $p \in X$.

[Hint: Use the (holomorphic) inverse function theorem, see e.g. [GH78], p. 18.]

(d) Deduce that L is basepoint free and φ is a closed embedding if $H^1(\mathcal{I}_p \otimes \mathcal{L}) = H^1(\mathcal{I}_p^2 \otimes \mathcal{L}) = 0$ for all $p \in X$ and $H^1(\mathcal{I}_{p,q} \otimes \mathcal{L}) = 0$ for all $p, q \in X$, $p \neq q$.

(5) Let X be a compact complex curve of genus g, and L a holomorphic line bundle on X.

(a) Show that $H^1(X, L) = 0$ for $\deg L > 2g - 2$.

[Hint: Recall Serre duality: For X a compact complex manifold of dimension n, $\omega_X = \Omega^n_X$ the canonical line bundle, and $E \to X$ a holomorphic vector bundle, we have a perfect pairing

$$H^k(X, \mathcal{E}) \times H^{n-k}(X, \omega_X \otimes \mathcal{E}^*) \to \mathbb{C}.$$

Now use Q3 and Q1b.]

(b) Show that if $\deg L > 2g$ then L defines a closed embedding $\varphi\colon X \to \mathbb{P}^m$ where $m = \deg L - g$.

[Hint: Use Q4d, part (a), and Riemann–Roch. Note that since $\dim X = 1$, $\mathcal{I}_p = \mathcal{O}_X(-p)$ is a line bundle, etc.]

(6) Recall the exact sequence

$$0 \to H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \to \operatorname{Pic} X \xrightarrow{c_1} H^{1,1} \cap H^2(X, \mathbb{Z}) \to 0$$

given by the long exact sequence of cohomology associated to the exponential sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0, \quad f \mapsto e^{2\pi i f}.$$

Let V be a complex vector space of dimension n, $\{\lambda_1, \ldots, \lambda_{2n}\}$ an $\mathbb{R}$-basis of V, $L = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_{2n} \subset V$ the lattice generated by $\lambda_1, \ldots, \lambda_{2n}$, and $X = V/L$ the associated complex torus of dimension n. (So in particular X is isomorphic to $(S^1)^{2n}$ as a C^∞ manifold.)

Note that $\pi_1(X) = H_1(X, \mathbb{Z}) = L$ and $H^k(X, \mathbb{Z}) = \wedge^k L^*$ by the Kunneth formula. In terms of de Rham cohomology, let $x_1, \ldots, x_{2n}$ be the real coordinates on V dual to $\lambda_1, \ldots, \lambda_{2n}$, then $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $i_1 < \cdots < i_k$ is a $\mathbb{Z}$-basis of $H^k(X, \mathbb{Z}) \subset H^k(X, \mathbb{R})$. (That is, the de Rham cohomology is identified with the translation invariant forms.)

Let $z_1, \ldots, z_n$ be complex coordinates on V, and consider the Hodge decomposition $H^k(X, \mathbb{C}) = \bigoplus H^{p,q}$ (recall that complex tori are Kähler). Then $H^{p,q} \subset H^k(X, \mathbb{C})$ has $\mathbb{C}$-basis $dz_{i_1} \wedge \cdots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$.

(a) Suppose that $n = 2$. Show that there is a set of countably many hypersurfaces in $V^{\oplus 4} \simeq \mathbb{C}^8$ such that $H^{1,1} \cap H^2(X, \mathbb{Z}) = 0$ iff $(\lambda_1, \ldots, \lambda_4)$ lies in the complement of the union of these hypersurfaces.

[Hint: Identify $V = \mathbb{C}^2$ using the complex coordinates z_1, z_2 and write $\lambda_j = (\lambda_{1j}, \lambda_{2j})$. Then $dz_i = \sum \lambda_{ij} dx_j$. Let $\omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j$, $a_{ij} \in \mathbb{Z}$ be an integral 2-form on X. Then ω is of type $(1,1)$ iff $dz_1 \wedge dz_2 \wedge \omega = 0$ (why?). Writing this equation in terms of the λ_{ij} gives a quadric hypersurface $q = 0$.]

(b) Assume $H^{1,1} \cap H^2(X, \mathbb{Z}) = 0$. Show that $\operatorname{Cl}(X) = 0$ and $\operatorname{Pic} X = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ (a complex torus of dimension n). In particular, there exists a holomorphic line bundle on X which does not admit a nonzero meromorphic section.

[Hint: A complex torus is Kähler. So if $Y \subset X$ is a prime divisor (irreducible analytic subset of codimension 1) then $0 \neq [Y] \in H_{2n-2}(X, \mathbb{Z})$.]

(7) Let X be a Kähler manifold of dimension n and $Z \subset X$ an irreducible analytic subset of dimension k. Let $[Z] \in H_{2k}(X, \mathbb{Z})$ be the associated homology class (the *fundamental class* of Z). Then the Poincaré dual cohomology class $\mathrm{PD}([Z]) \in H^{2n-2k}(X, \mathbb{Z})$ has type $(n-k, n-k)$. To see this, recall that the Poincaré duality perfect pairing is given in de Rham cohomology by

$$H^{2k}(X, \mathbb{R}) \times H^{2n-2k}(X, \mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta,$$

and (extending scalars from $\mathbb{R}$ to $\mathbb{C}$) via the Hodge decomposition this decomposes as a direct sum of perfect pairings

$$H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{C}$$

where $p + q = 2k$. Now, by definition

$$\int_X \alpha \wedge \mathrm{PD}([Z]) = \int_Z \alpha$$

which vanishes for α of type $(p, q) \neq (k, k)$. It follows that $\mathrm{PD}([Z])$ has type $(n-k, n-k)$.

Use this fact to describe the Hodge diamond of $\mathbb{P}^n$, and so give another proof that $H^q(\mathcal{O}_{\mathbb{P}^n}) = 0$ for $q > 0$.

(8) Let X and Y be complex manifolds and $X \times Y$ the Cartesian product with projections $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$. We have a group homomorphism

$$\operatorname{Pic} X \oplus \operatorname{Pic} Y \to \operatorname{Pic}(X \times Y), \quad (L, M) \to p_1^* L \otimes p_2^* M. \qquad (*)$$

(a) Show that if $X = Y = \mathbb{P}^1$ then $(*)$ is an isomorphism.

[Hint: (1) Use the Kunneth formula to compute the cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$. (Note: The Kunneth formula holds with $\mathbb{Z}$ coefficients if the cohomology of X and Y is torsion-free.) (2) Determine the Hodge diamond of $\mathbb{P}^1 \times \mathbb{P}^1$ and deduce that $c_1 \colon \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \to H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ is an isomorphism (cf. Q7).]

(b) Show that if $X = Y = E$ is an elliptic curve (a complex curve of genus 1) then $(*)$ is *not* an isomorphism.

[Hint: Let $p_0 \in E$ be a choice of base point. Recall that E has the structure of an abelian group with identity p_0 (because there is an isomorphism $E \simeq \mathbb{C}/\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ with $p_0 \mapsto 0$.) Show that $F_1 = \{p_0\} \times E$, $F_2 = E \times \{p_0\}$, and the diagonal $\Delta \subset E \times E$ are linearly independent in $H_2(E \times E, \mathbb{Z})$ by e.g. computing intersection products. (Note that F_i is a fiber of p_i and Δ is a fiber of the map $E \times E \to E$, $(p, q) \mapsto p - q$ (using the group law on E). So $[F_1]^2 = [F_2]^2 = [\Delta]^2 = 0$ (why?).) Deduce that $(*)$ is *not* surjective.]

(9) Recall that we say a compact complex manifold X is *Fano* if ω_X^* is ample, where $\omega_X = \wedge^n \Omega_X$ is the canonical line bundle. Show that if X is Fano then $c_1 \colon \operatorname{Pic} X \to H^2(X, \mathbb{Z})$ is an isomorphism.

[Hint: Recall the Kodaira vanishing theorem: If X is a compact complex manifold and $L \to X$ is an ample line bundle then $H^k(\omega_X \otimes L) = 0$ for $k > 0$.]

References

[GH78] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley, 1978.