Math 797AS Homework 1

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(1) Let X be a complex manifold. We can forget the complex structure and consider the underlying smooth manifold. Show that the complex charts of X determine an orientation of the underlying smooth manifold.

[Hint: Consider the transition map between two charts with coordinates $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$, $j = 1, \ldots, n$. Let $B \in \mathrm{GL}_{2n}(\mathbb{R})$ be the matrix of the real derivative of the transition map at a point with respect to the real bases $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$ and $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}$ of the tangent spaces. Now change bases (after extending scalars from $\mathbb{R}$ to $\mathbb{C}$) to $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}$ and $\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_n}, \frac{\partial}{\partial \bar{w}_1}, \ldots, \frac{\partial}{\partial \bar{w}_n}$. Show that the matrix with respect to these bases is the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$ where $A = (\frac{\partial w_j}{\partial z_k})$ is the matrix of the complex derivative of the transition map with respect to the complex bases $\frac{\partial}{\partial z_1}, \ldots \frac{\partial}{\partial z_n}$ and $\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_n}$ of the tangent spaces. Deduce that $\det B = |\det A|^2 > 0$.]

(2) Let $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$ be the complex projective n-space, where $\mathbb{C}^\times$ acts by scalar multiplication. Consider the sphere

$$S^{2n+1} = \{(z_0, \ldots, z_n) \mid \sum |z_j|^2 = 1\} \subset \mathbb{C}^{n+1}$$

and the induced action of $\mathrm{U}(1) = \{z \mid |z| = 1\} \subset \mathbb{C}^\times$ on S^{2n+1}. Show that $\mathbb{P}^n = S^{2n+1}/\,\mathrm{U}(1)$ and deduce that $\mathbb{P}^n$ is compact.

(3) A complex curve (or Riemann surface) of genus 1 is isomorphic to a complex torus $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ is a basis of $\mathbb{C}$ regarded as an $\mathbb{R}$-vector space (this is an instance of the Riemann uniformization theorem). Show that a morphism (holomorphic

map) of complex manifolds $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is induced by an affine transformation $z \mapsto \alpha z + \beta$, for some $\alpha, \beta \in \mathbb{C}$. Deduce that the *moduli space* parametrizing isomorphism types of complex curves of genus 1 is identified with the quotient of the upper half plane

$$\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$$

by the action of $\mathrm{SL}(2, \mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Remark: In general, for $g \geq 2$, the moduli space M_g parametrizing isomorphism types of complex curves of genus g is a complex orbifold of dimension $3g - 3$.

(4) Let X_1 and X_2 be compact, oriented, simply connected, smooth 4-manifolds. Show that the connected sum $X = X_1 \# X_2$ is a compact, oriented, simply connected smooth 4 manifold, such that $H_2(X, \mathbb{Z}) = H_2(X_1, \mathbb{Z}) \oplus H_2(X_2, \mathbb{Z})$ and the intersection product $Q_X = Q_{X_1} \oplus Q_{X_2}$.

[Hint: Use the Van Kampen theorem and the Mayer–Vietoris sequence. See e.g. Hatcher.]

(5) Let M be an compact oriented manifold such that $d = \dim_{\mathbb{R}} M$ is odd. Prove that the Euler number

$$e(M) = \sum_{i=0}^{d} (-1)^i \dim_{\mathbb{R}} H_i(M, \mathbb{R})$$

equals zero.

(6) Recall in class we described a rational elliptic surface obtained by blowing up the 9 intersection points of two general cubic curves $C_0 = (F = 0)$ and $C_\infty = (G = 0)$ in $\mathbb{P}^2$. For F and G general, every element of the *pencil* of cubic curves $C_{(\lambda:\, \mu)} = (\lambda F + \mu G = 0) \subset \mathbb{P}^2$, $(\lambda : \mu) \in \mathbb{P}^1$ is either smooth or has a unique singularity which is a *node*, that is, in local coordinates at the singular point $p \in C = C_{(\lambda:\, \mu)}$, we have an isomorphism of germs

$$(p \in C \subset \mathbb{P}^2) \simeq (0 \in (z_1 z_2 = 0) \subset \mathbb{C}^2_{z_1, z_2})$$

(this is a special case of a lemma of Lefschetz: $\{C_t\}_{t\in\mathbb{P}^1}$ is a so-called *Lefschetz pencil*). In the singular case C is topologically a *pinched torus* obtained from $T^2 = S^1 \times S^1$ by collapsing a curve $S^1 \times \{q\}$ to a point.

Now consider the associated elliptic fibration $f\colon X \to \mathbb{P}^1$, with fibers $f^{-1}(t) = C_t$. Show that there are exactly 12 singular fibers by computing the Euler number of X in two ways: first using the description as a blowup of $\mathbb{P}^2$, and second in terms of the elliptic fibration.

[Hints:(0) By Mayer–Vietoris $e(X \cup Y) = e(X) + e(Y) - e(X \cap Y)$. (1) If $\pi\colon E \to B$ is a locally trivial fiber bundle with fiber F then $e(E) = e(B)e(F)$. (2) If $C = f^{-1}(p) \subset X$ is a singular fiber and $p \in U \subset \mathbb{P}^1$ is a small open disc centered at p with closure $\overline{U}$ then $N = f^{-1}(\overline{U})$ is a manifold with boundary such that $C = f^{-1}(p) \subset N$ is a deformation retract.]

(7) Recall the construction of the logarithmic transform for an elliptic fibration (cf. Griffiths and Harris, p. 565–567): Let $f\colon X \to C$ be a holomorphic map from a complex surface X to a complex curve C such that a general fiber $F = f^{-1}(p)$ of f is a (smooth) complex curve of genus 1. Let $p \in U \subset C$ be a small open disc centered at p, and identify $f^{-1}(U) \to U$ with

$$g\colon Y := \mathbb{C}_z \times \mathbb{D}_t/\mathbb{Z}^2 \to \mathbb{D}_t$$

where $\mathbb{D}_t = \{t \mid |t| < 1\}$ and the group action is given by

$$(a,b)\colon (z,t) \mapsto (z + a + b\tau(t), t)$$

where $\tau\colon \mathbb{D}_t \to \mathbb{C}_z$ is holomorphic and $\operatorname{Im}\tau(t) \neq 0$ for all t. Fix $m \in \mathbb{N}$ and $k \in \mathbb{Z}/m\mathbb{Z}$ such that $(k,m) = 1$.

Let $Z = Y \times_{\mathbb{D}_t} \mathbb{D}_s \to \mathbb{D}_s$ be the pullback of the family $Y \to \mathbb{D}_t$ via $\mathbb{D}_s \to \mathbb{D}_t$, $s \mapsto s^m$. So

$$Z = \mathbb{C}_w \times \mathbb{D}_s/\mathbb{Z}^2 \to \mathbb{D}_s$$

where the action is given by

$$(a,b)\colon (w,s) \mapsto (w + a + b\tau(s^m), s).$$

Let $g'\colon Y' \to \mathbb{D}_t$ be the quotient of $Z \to \mathbb{D}_s$ by the $\mathbb{Z}/m\mathbb{Z}$ action given by

$$\mathbb{Z}/m\mathbb{Z} \ni 1\colon (w,s) \mapsto (w + k/m, e^{2\pi i/m} \cdot s).$$

(a) Show that there is an isomorphism $(Y')^\times \to Y^\times$ of the restriction of the families to the punctured disc $\mathbb{D}_t^\times = \mathbb{D}_t \setminus \{0\}$ given by

$$(w, s) \mapsto (w - \frac{k}{2\pi i} \log s, s^m).$$

So we can glue $Y' \to \mathbb{D}$ to $X \setminus F \to C \setminus \{p\}$ along $Y^\times \to \mathbb{D}^\times$ to obtain a new elliptic fibration $f' \colon X' \to C$.

(b) Show that the fiber $F' = g'^{-1}(0)$ of $g' \colon Y' \to \mathbb{D}_t$ over $0 \in \mathbb{D}_t$ is a smooth fiber of multiplicity m, that is, near a point of F' there are local coordinates (z_1, z_2) on Y' such that the map g' is given by $(z_1, z_2) \mapsto z_2^m$. So the logarithmic transform replaces a smooth fiber of multiplicity 1 with a smooth fiber of multiplicity m.

(8) Recall the Hopf surface $X = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$, where the action is given by

$$(z_1, z_2) \mapsto \frac{1}{2}(z_1, z_2).$$

Show that there is an elliptic fibration $X \to \mathbb{P}^1$ such that all the fibers are isomorphic.

[Hint: There is an isomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^\times$ defined by $z \mapsto \exp(2\pi i z)$.]

(9) (Optional) Study the construction of symplectic and Kähler quotients in [HKLR87], §3A,B,C, and work it out explicitly in the case of complex projective space $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times = S^{2n+1}/S^1$ to obtain the Fubini Study metric, following our discussion in class.

(10) (Optional) Study the construction of a complex structure on $S^3 \times S^3$ in [CE53]. See also Wikipedia. This gives a simply connected compact complex manifold X which is not Kähler (because $H^2(X, \mathbb{R}) = 0$). (Remark: Complex structures on $S^3 \times S^3$ arise in the conjecture of Miles Reid on Calabi–Yau 3-folds, see [R87].)

(11) (Optional) Study the construction of a non-Kähler compact complex 3-fold X by Hironaka, see e.g [H77], p. 444, Example 3.4.2 (cf. Example 3.4.1). These examples have the property that the field of meromorphic functions on X has transcendence degree over $\mathbb{C}$ equal to the complex dimension of X (as for a projective variety). They are not Kähler because there is a complex curve $C \subset X$ such that the homology class of C in $H_2(X, \mathbb{Z})$ is equal to zero.

References

[HKLR87] N. Hitchin, A. Karlhede, U. Lindström, M. Rocek, Hyper-Kähler metrics and supersymmetry, Comm. Math. Phys. 108 (1987), no. 4, 535–589.

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[R87] M. Reid, The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible, Math. Ann. 278 (1987), no. 1-4, 329–334.