Math 461 Midterm review problems

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(1) A *rhombus* is a 4-sided polygon such that all the sides have the same length.

(a) Prove that the diagonals of a rhombus meet at right angles.

(b) Prove that the opposite sides of a rhombus are parallel.

(2) Let ΔABC be a triangle such that $\angle ACB = \pi/2$. Let L be the line through C perpendicular to AB. Let D be the intersection point of L and AB. Prove that $|AD| \cdot |DB| = |CD|^2$.

(3) Let A, B, C, D be points on a circle and suppose that the chords AB and CD meet at a point P inside the circle. Prove that $|AP||BP| = |CP||DP|$.

(4) Suppose given a circle $\mathcal{C}$. Give a ruler and compass construction of a rectangle $ABCD$ such that the vertices A, B, C, D lie on the circle and the angle between the diagonals is $\pi/6$.

[Note: For this problem and the other ruler and compass problems below, you may use the ruler and compass constructions described in class and in the text book (perpendicular bisector of a line segment, angle bisector, etc.) as components of your construction.]

(5) Suppose given a circle $\mathcal{C}$. Give a ruler and compass construction of an isosceles right-angled triangle ΔABC such that the sides are tangent to the circle.

(6) Suppose given an angle θ and two lengths l and h. Give a ruler and compass construction of a triangle ABC with $\angle CAB = \theta$, $|BC| = l$,

and perpendicular height h from A to BC. (Here we assume that $\tan(\theta/2) \leq l/2h$, otherwise such a triangle does not exist (why?).)

[Hint: First construct a triangle $A'BC$ with $\angle CA'B = \theta$ and $|BC| = l$. What is the locus of points A such that $\angle CAB = \theta$?]

(7) Find the intersection points of the circle with center the origin and radius 3 and the circle with center $(1, 2)$ and radius 2.

(8) Find the center and radius of the circle passing through the points $A = (0, 0)$, $B = (1, 2)$, and $C = (-1, 3)$.

(9) Show that the locus of points in the plane equidistant from a line L and a point P not on L is a parabola.

[Hint: Without loss of generality, we can choose coordinates such that the line L is the x-axis and the point P has coordinates $(0, b)$ for some $b \neq 0$.]

(10) Suppose given a triangle ΔABC such that $\angle ABC$ and $\angle BAC$ are acute (less than $\pi/2$). What is the maximum area of a rectangle $DEFG$ with base DE on the side AB, vertex F on the side BC, and vertex G on the side CA?

(11) Let L be a line in $\mathbb{R}^2$. Recall that Refl_L denotes reflection in the line L, an isometry of $\mathbb{R}^2$.

Let P, Q be two points in $\mathbb{R}^2$ on the same side as L. Suppose we place a mirror on the line L (standing vertically, perpendicular to the plane), with the silvered side pointing towards P and Q. Show that the perceived position of P for an observer at Q looking in the mirror is $\text{Refl}_L(P)$. (In other words, light from the object at P reflected in the mirror travels the same distance to Q and arrives in the same direction as light from an object at $\text{Refl}_L(P)$ when the mirror is removed.)

[Hint: For reflection of light rays in a mirror, the incoming ray makes the same angle with the mirror as the outgoing ray ("angle of incidence equals angle of reflection").]

(12) Give a precise geometric description for each of the following isometries $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ as a translation, rotation, reflection, or glide reflection.

(a) $T(x, y) = (x + 5, y + 7)$.

(b) $T(x,y) = (1-y, 3-x)$.

(c) $T(x,y) = (y+2, 8-x)$.

(d) $T(x,y) = \frac{1}{13}(5x+12y+4, 12x-5y-6)$.

(13) Give an algebraic formula for the rotation about the point $(4,2)$ through angle $\pi/4$ radians counterclockwise.

(14) (a) Let L be a line through the origin in $\mathbb{R}^2$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in L. Let $\mathbf{n}$ be a vector perpendicular to L. Show that T is given by the formula

$$T(\mathbf{x}) = \mathbf{x} - 2\left(\frac{\mathbf{x}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}.$$

[Hint: Here $\mathbf{u}\cdot\mathbf{v}$ is the dot product, defined by $(u_1,u_2)\cdot(v_1,v_2) = u_1v_1 + u_2v_2$. In particular $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2$, where $\|\mathbf{u}\|$ is the length of the vector $\mathbf{u}$. Recall that we have the formula

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.]

(b) Find an algebraic formula for the reflection in the line $y = 3x$.

(c) Find an algebraic formula for the reflection in the line $y = 3x+1$.

(15) Give a precise geometric description for each of the following isometries as a translation, rotation, reflection, or glide reflection.

(a) The composition of reflection in the line $x = 3$ followed by reflection in the line $y = \sqrt{3}x$.

(b) The composition of reflection in the line $y = 2x+1$ followed by reflection in the line $y = 2x+5$.

(c) The composition of rotation about the point $(1,1)$ through angle $\pi/2$ counterclockwise followed by rotation about the point $(2,2)$ through angle $\pi/2$ counterclockwise.

(d) The composition of rotation about the point $(3,2)$ through angle $\pi/3$ counterclockwise followed by translation by $(4,0)$

(16) Let L and M be two distinct lines in the plane. Show that $\mathrm{Refl}_M \circ \mathrm{Refl}_L = \mathrm{Refl}_L \circ \mathrm{Refl}_M$ iff L and M are perpendicular.

(17) Recall that $\mathrm{Rot}(P, \theta)$ denotes rotation about a point P through angle θ counterclockwise. Let $ABCD$ be a square such that the vertices A, B, C, D are in counterclockwise order. Determine the composition

$$\mathrm{Rot}(A, \pi/2) \circ \mathrm{Rot}(B, \pi/2) \circ \mathrm{Rot}(C, \pi/2) \circ \mathrm{Rot}(D, \pi/2).$$

Justify your answer carefully.

(18) Express the isometry $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 6, 4 - y)$ as a composite of at most 3 reflections.

(19) Let $A = (0, 0), B = (1, 0), C = (0, 2)$, and $A' = (3, 5), B' = (3, 4), C' = (1, 5)$. Find an isometry T sending ΔABC to $\Delta A'B'C'$. (Give a geometric description and an explicit algebraic formula.)