# 411 Midterm 2 Review Questions

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(1) Recall that the *symmetric group* $S_n$ is the group of permutations of the set $\{1, 2, \ldots, n\}$, with operation given by composition of functions. Let $\sigma \in S_8$ be the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 1 & 8 & 3 & 7 & 4 & 2 \end{pmatrix}.$$

(a) Express $\sigma$ as a product of disjoint cycles.

(b) What is the order of $\sigma$?

(c) Is $\sigma$ even or odd?

(d) Express $\sigma$ as a product of transpositions.

(2) Repeat Q1 for the permutation $\sigma \in S_9$ given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 3 & 6 & 9 & 2 & 1 & 5 & 4 \end{pmatrix}.$$

(3) Let $\sigma \in S_n$ be a permutation. Then we can write $\sigma$ as a product $\sigma = \tau_1\tau_2\cdots\tau_r$ of some number $r$ of disjoint cycles $\tau_1, \ldots, \tau_r$. Let $l_i$ be the length of the cycle $\tau_i$ for $i = 1, \ldots, r$.

(a) What is the order of $\sigma$?

(b) Show that $\sigma$ can be written as a product of $(l_1 - 1) + (l_2 - 1) + \cdots + (l_r - 1)$ transpositions.

(4) Give the definition of even and odd permutations. Define the alternating group $A_n$. What is the order of $S_n$? What is the order of $A_n$?

(5) Let $H$ be the subgroup $\langle(13)\rangle = \{e, (13)\}$ of $S_3$. Find the left cosets of $H$ in $S_3$ and the right cosets of $H$ in $S_3$. Are the left cosets the same as the right cosets?

(6) (a) List the elements of $A_4$.

(b) Let $H$ be the cyclic subgroup of $A_4$ generated by the 3-cycle $(123)$. Find the left cosets of $H$ in $A_4$. Are the left cosets the same as the right cosets?

(7) Recall that for a subgroup $H$ of a group $G$, the *index* of $H$ in $G$ is the number of left cosets of $H$ in $G$. Let $G = \mathbb{Z} \times \mathbb{Z}$ and let $H$ be the subgroup of $G$ consisting of pairs $(a, b)$ such that $a + b$ is divisible by 3. What is the index of $H$ in $G$?

(8) (a) State Lagrange's theorem.

(b) Let $G$ be a finite group such that $|G|$ is prime. Show that $G$ is cyclic.

(9) Find all abelian groups of order 600 up to isomorphism.

(10) Determine whether the groups $\mathbb{Z}_{24} \times \mathbb{Z}_{90} \times \mathbb{Z}_{100}$ and $\mathbb{Z}_{36} \times \mathbb{Z}_{40} \times \mathbb{Z}_{150}$ are isomorphic.

(11) Compute the order of the element $(3, 4, 5) \in \mathbb{Z}_{12} \times \mathbb{Z}_{10} \times \mathbb{Z}_{13}$.

(12) Let $p$ be a prime. How many elements of order $p$ are there in $\mathbb{Z}_{p^2}$? How many elements of order $p$ are there in $\mathbb{Z}_p \times \mathbb{Z}_p$?

(13) For $n \geq 3$ the dihedral group $D_n$ is the group of symmetries of a regular $n$-gon (a polygon with $n$ sides of equal length).

(a) What is the order of $D_n$?

(b) Let $\rho \in D_n$ be the rotation about the center $p$ of the polygon through an angle $\theta = 2\pi/n$ anticlockwise. Let $\mu \in D_n$ be a reflection. Express all the elements of $D_n$ in terms of $\rho$ and $\mu$. [Hint: What are the cosets of $H = \langle\rho\rangle$ in $D_n$?]

(14) Show that $D_6$ is isomorphic to the subgroup of the symmetric group $S_6$ generated by the elements $(123456)$ and $(26)(35)$. [Hint: label the vertices of the hexagon by $1, 2, 3, 4, 5, 6$.] Use this to prove that $\mu\rho =$

$\rho^{-1}\mu$ in $D_6$ where $\mu$ and $\rho$ are defined as above. [This was proved in class for $D_n$ by another method.]

(15) (a) Show that every proper subgroup of the dihedral group $D_7$ is cyclic.

(b) Give an example of a proper subgroup of $D_4$ that is not cyclic.

(16) What is the maximum possible order of an element of the following groups?

(a) $\mathbb{Z}_n$.

(b) $\mathbb{Z}_{12} \times \mathbb{Z}_{15} \times \mathbb{Z}_{63}$.

(c) $S_9$

(d) $A_9$

(e) $D_9$

(17) Let $G_1$ and $G_2$ be groups. Show carefully that $G_1 \times G_2$ and $G_2 \times G_1$ are isomorphic.

(18) Let $G$ be a finite group such that $a^2 = e$ for every $a \in G$.

(a) Show that $G$ is abelian

(b) Show that $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (where there are some number $s$ of copies of $\mathbb{Z}_2$).

(19) Let $G_1$ and $G_2$ be groups. Show that if $G_1 \times G_2$ is cyclic then both $G_1$ and $G_2$ are finite cyclic groups and their orders are coprime.