Math 411 Midterm 2, Thursday 11/17/11, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 7 questions for a total of 75 points. Calculators, notes, and textbook are not allowed. Justify all your answers carefully. If you use a result proved in the textbook or class notes, state the result precisely.

Q1. (10 points) Let $\sigma \in S_9$ be the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 7 & 2 & 3 & 9 & 1 & 4 & 6 \end{pmatrix}$$

(a) (2 points) Write σ as a product of disjoint cycles.

(b) (2 points) What is the order of σ?

(c) (2 points) Write σ as a product of transpositions.

(d) (4 points) Compute σ^{100}.

Q2. (8 points)

(a) (3 points) Find an element of S_5 of order 6 or prove that no such element exists.

(b) (5 points) Find an element of S_6 of order 7 or prove that no such element exists.

Q3. (12 points)

(a) (4 points) List all the elements of the alternating group A_4.

(b) (8 points) Find the left cosets of the subgroup H of A_4 given by

$$H = \{e, (12)(34), (13)(24), (14)(23)\}.$$

Q4. (10 points)

(a) (3 points) State Lagrange's theorem.

(b) (7 points) Let p be a prime, $p \geq 3$. Let D_p be the dihedral group of symmetries of a regular p-gon (a polygon with p sides of equal length). What are the possible orders of subgroups of D_p? Give an example in each case.

Q5. (13 points)

(a) (3 points) State a theorem which describes all finitely generated abelian groups.

(b) (5 points) List all abelian groups of order 108 up to isomorphism.

(c) (5 points) Determine whether the groups $\mathbb{Z}_{20} \times \mathbb{Z}_{24} \times \mathbb{Z}_{75}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{40} \times \mathbb{Z}_{300}$ are isomorphic.

Q6. (10 points) Consider the symmetric group S_n.

(a) (2 points) Let i, j, k be distinct elements of $\{1, 2, \ldots, n\}$. Show that the product $(ij)(ik)$ is a 3-cycle (a cycle of length 3).

(b) (4 points) Let i, j, k, l be distinct elements of $\{1, 2, \ldots, n\}$. Show that the product $(ij)(kl)$ can be written as a product of two 3-cycles. [Hint: $(ij)(kl)$ can be rewritten as $(ij)(ik)^2(kl)$.]

(c) (4 points) Show that every even permutation $\sigma \in A_n$ can be written as a product of 3-cycles.

Q7. (12 points) Let $H = \langle \sigma, \tau \rangle$ be the subgroup of S_6 generated by $\sigma = (1234)$ and $\tau = (56)$. (That is, H is the smallest subgroup of S_6 containing σ and τ.)

(a) (4 points) List the elements of H.

(b) (4 points) Show that H is abelian.

(c) (4 points) Describe an isomorphism $\phi\colon G \to H$ where G is a direct product of cyclic groups (to be determined).