411 Midterm 1 Review Questions

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(1) Give the definition of a group G.

(2) Suppose G is a group and x, y, z, w are elements of G satisfying the equation $xyz^{-1}w = e$. Solve for y.

(3) In each of the following cases, determine whether the given set with binary operation is a group. Explain.

(a) The set $S = \{x \in \mathbb{Z} \mid x \geq 0\}$ with operation given by addition.

(b) The set of all real 2×2 matrices with operation matrix multiplication.

(c) The set

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

with operation given by matrix multiplication.

(d) The set $S = \mathbb{R} \backslash \{-1\}$ of all real numbers except -1 with operation $*$ defined by $a * b = ab + a + b$.

(e) The set of continuous functions $f\colon [0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$, and f is increasing (that is, $f(a) < f(b)$ for $a < b$), with operation given by composition of functions.

(4) Let G be the set $\{1, 2, 3, 4, 5, 6\}$ with binary operation $*$ given by the

table below. Is G a group? Explain.

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	1	6	4	2
4	4	6	5	1	2	3
5	5	3	6	2	1	4
6	6	4	2	5	3	1

(5) Let G and G' be groups. Give the definition of an isomorphism $\phi\colon G \to G'$.

(6) Let $\phi\colon G \to G'$ be an isomorphism of groups. Show that $\phi^{-1}\colon G' \to G$ is an isomorphism.

(7) Let G be a group and $\phi\colon G \to G$ the function defined by $\phi(a) = a^{-1}$. Is ϕ an isomorphism? Explain.

(8) For each of the groups G defined in Q3(c) and Q3(d), describe an isomorphism $\phi\colon G \to G'$ where G' is a well-known group.

(9) Give the definition of a subgroup H of a group G.

(10) List all the subgroups of the following groups.

(a) $(\mathbb{Z}, +)$.

(b) $(\mathbb{Z}_{18}, +)$.

(c) $(\mathbb{Z}_n, +)$, where $n \in \mathbb{Z}^+$ is a positive integer.

(d) The group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$ with operation vector addition modulo 2. (This group is sometimes called the *Klein 4-group.*)

(e) The group G of symmetries of an equilateral triangle.

(11) In each of the following cases, determine whether the given subset H of the group G is a subgroup. Explain.

(a) Let G be the group of symmetries of an equilateral triangle and H the subset of reflections.

(b) Let

$$G = \mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0 \right\}$$

be the group of 2×2 invertible matrices with operation given by matrix multiplication. Let H be the subset of G given by

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R}, \quad a \neq 0 \right\}.$$

(c) Let $G = \mathbb{C}^\times$ be the group of nonzero complex numbers with operation given by multiplication of complex numbers. Let H be the subset of G given by

$$H = \{z \in \mathbb{C} \mid |z| = 1\}.$$

(d) Let $G = \mathrm{GL}_2(\mathbb{R})$ and let H be the subset of G given by matrices having determinant $+1$ or -1, that is,

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R}, \quad ad - bc = \pm 1 \right\}.$$

(12) Let H and K be subgroups of a group G. Show that the intersection $H \cap K$ is a subgroup of G.

(13) Prove or give a counterexample: If H and K are subgroups of a group G then the union $H \cup K$ is a subgroup of G.

(14) Give the definition of a cyclic group. (Make sure you understand the multiplicative notation a^n and additive notation $n \cdot a$ for a an element of a group G and $n \in \mathbb{Z}$.)

(15) Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$ denote the group of nth roots of unity, with operation given by multiplication of complex numbers. Show that U_n is a cyclic group of order n and identify a generator.

(16) Which of the following groups are cyclic? Explain.

(a) $(\mathbb{Z}, +)$.

(b) The group $\mathbb{Z}_7^\times = \{1,2,3,4,5,6\}$ with operation multiplication modulo 7.

(c) The group

$$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$$

with operation given by matrix multiplication.

(d) The Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(e) The group G of symmetries of an equilateral triangle.

(17) List all the generators of the cyclic group $(\mathbb{Z}_{20}, +)$. What are the generators of $(\mathbb{Z}_n, +)$ for $n \in \mathbb{Z}^+$?

(18) List all the generators of U_6. List all the subgroups of U_6.

(19) Give the definition of the order of an element a of a group G.

(20) Let G be a group and $a \in G$ an element of order $n \in \mathbb{Z}^+$. What is the order of a^m for $m \in \mathbb{Z}^+$?

(21) Let G be a group and $a, b \in G$ elements of G. Show that the order of bab^{-1} is equal to the order of a.

(22) Give the definition of the cyclic subgroup $H = \langle a \rangle$ of a group G generated by an element a. Explain why the order of the group $\langle a \rangle$ is equal to the order of the element a.

(23) Let $G = \mathrm{GL}_2(\mathbb{R})$ be the group of real 2×2 invertible matrices with operation given by matrix multiplication. Compute explicitly the cyclic subgroup of G generated by the element $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.