Math 411 Midterm 1, Thursday 10/13/11, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 6 questions for a total of 70 points. Calculators, notes, and textbook are not allowed. Justify all your answers carefully. If you use a result proved in the textbook or class notes, state the result precisely.

Q1 (16 points). Give the definition of a group (3 points). For each of the following cases, determine whether the given set with binary operation is a group.

(a) (3 points) The set $S = \{2^n \mid n \in \mathbb{Z}\}$ with operation given by multiplication.

(b) (3 points) The set $S = \mathbb{R}$ with operation $*$ given by $a * b = \max(a, b)$ (that is, $a * b = a$ if $a \geq b$ and $a * b = b$ if $a \leq b$).

(c) (3 points) The set $S = \{e, a, b\}$ with operation $*$ defined by the table

$*$	e	a	b
e	e	a	b
a	a	e	a
b	b	a	e

(d) (4 points) The set

$$S = \{f \colon \mathbb{R} \to \mathbb{R} \mid f(x) = ax + b, \quad a, b \in \mathbb{R}, \quad a \neq 0\}$$

with operation given by composition of functions.

Q2 (13 points). Give the definition of a subgroup of a group (3 points). For each of the following cases, determine whether the subset H of the group G is a subgroup.

(a) (3 points) Let G be the group of symmetries of an equilateral triangle and H the subset of rotations.

(b) (3 points) Let $G = \mathbb{C}^\times$ be the group of non-zero complex numbers with operation given by multiplication of complex numbers. Let $H \subset G$ be the subset given by $H = \{z \in G \mid |z| \leq 1\}$.

(c) (4 points) Let

$$G = \mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0 \right\}$$

be the group of 2×2 invertible matrices with operation given by matrix multiplication. Let $H \subset G$ be the subset of upper triangular matrices, that is,

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \;\middle|\; c = 0 \right\}.$$

Q3 (10 points). Let G be a group. Let H be the subset of G given by

$$H = \{a \in G \mid a^2 = e\}.$$

(a) (6 points) Show that if G is abelian then H is a subgroup of G.

(b) (4 points) Give an example of a group G such that the subset H defined above is not a subgroup of G.

Q4 (12 points).

(a) (3 points) Give the definition of an isomorphism of groups $\phi\colon G \to G'$.

(b) (3 points) Let $G = \mathbb{R}$ be the group of real numbers with operation addition, and $G' = \mathbb{R}^+$ the group of positive real numbers with operation multiplication. Show that the function $\phi\colon G \to G'$ given by $\phi(x) = e^x$ is an isomorphism of groups.

(c) (6 points) Let $G = \mathrm{GL}_2(\mathbb{R})$ be the group of 2×2 invertible matrices and let $A \in G$ be the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Compute the cyclic subgroup H of G generated by A. Describe an isomorphism $\phi\colon \mathbb{Z}_n \to H$ for some $n \in \mathbb{Z}^+$ (to be determined).

Q5 (9 points). Let $G = \mathbb{Z}_{12}$ be the group of integers modulo 12 with operation addition.

(a) (3 points) Find all the generators of G.

(b) (6 points) Find all the subgroups of G and draw the subgroup diagram showing inclusions of subgroups.

Q6 (10 points). Let H and K be subgroups of a group G.

(a) (3 points) Show that $H \cap K$ is a subgroup of G.

(b) (7 points) Suppose that H and K are cyclic of orders m and n, and $\gcd(m, n) = 1$. Show that $H \cap K = \{e\}$.