

411 Final Review Questions

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- (1) Let G and G' be groups. Recall that a *homomorphism* from G to G' is a function $\phi: G \rightarrow G'$ such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$. Which of the following functions are homomorphisms? If ϕ is a homomorphism, determine the kernel and image of ϕ .
- (a) $\phi: \mathbb{Z}_5 \rightarrow \mathbb{Z}, \phi(k) = k$.
 - (b) $\phi: \mathbb{R} \rightarrow U, \phi(\theta) = e^{i\theta}$. [Here $U = \{z \in \mathbb{C} \mid |z| = 1\}$ with operation multiplication of complex numbers.]
 - (c) $\phi: S_n \rightarrow \mathbb{Z}_2, \phi(\sigma) = 0$ if σ is even and $\phi(\sigma) = 1$ if σ is odd.
 - (d) G is an abelian group and $\phi: G \rightarrow G, \phi(a) = a^3$. [What happens if G is not abelian, for example $G = S_3$?]
 - (e) $\phi: \mathbb{Z}_7 \rightarrow \mathbb{Z}_3, \phi(k) = k \bmod 3$ (that is, $\phi(k)$ is the remainder on dividing k by 3.)
 - (f) $\phi: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_4, \phi(k) = k \bmod 4$.
 - (g) $\phi: \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}, \phi(k) = 6k \bmod 10$.
 - (h) $\phi: D_n \rightarrow \mathbb{Z}_2, \phi(g) = 0$ if g is a rotation (including the identity) and $\phi(g) = 1$ if g is a reflection.
- (2) Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R}, ac \neq 0 \right\}$$

be the group of 2×2 invertible upper triangular matrices with operation matrix multiplication.

- (a) Show that the function

$$\phi: G \rightarrow \mathbb{R}^\times, \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a$$

is a homomorphism from G to the group \mathbb{R}^\times of nonzero real numbers (with operation multiplication of real numbers).

- (b) Describe the kernel and image of ϕ .

- (3) Let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

be the group of 3×3 upper triangular matrices with 1's on the diagonal, with operation matrix multiplication.

- (a) Check that G is a group.

- (b) Show that the function

$$\phi: G \rightarrow \mathbb{R}, \quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto a.$$

is a homomorphism from G to the group of real numbers (with operation addition of real numbers).

- (c) Describe the kernel and image of ϕ .

- (4) (a) Let G and G' be finite groups such that $|G| = |G'|$ and $\phi: G \rightarrow G'$ a homomorphism from G to G' . Show that ϕ is an isomorphism iff $\ker \phi = \{e\}$. [Hint: The *pigeonhole principle* says that if G and G' are finite sets such that $|G| = |G'|$ and $\phi: G \rightarrow G'$ is a function from G to G' then ϕ is injective $\iff \phi$ is surjective $\iff \phi$ is bijective.]

- (b) Define a function $\phi: \mathbb{Z}_{mn} \rightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ by

$$k \mapsto (k \bmod m, k \bmod n).$$

Show that ϕ is a homomorphism, and compute the kernel of ϕ . Deduce that ϕ is an isomorphism iff $\gcd(m, n) = 1$ [Hint: Use part (a)].

- (5) Let $\phi: G \rightarrow G'$ be a homomorphism of groups with kernel H . Let $a \in G$ be an element of G and $b = \phi(a)$. Give a description of the set $\phi^{-1}(b) := \{x \in G \mid \phi(x) = b\}$ in terms of H .
- (6) Let $\phi: G \rightarrow G'$ be a homomorphism. Let a be an element of G . Show that the order of $\phi(a)$ in G' divides the order of a in G . [Recall that the *order* of an element $a \in G$ is the smallest positive integer n such that $a^n = e$. Note that $a^m = e$ iff m is divisible by the order of a .]
- (7) (a) Let $\phi: \mathbb{Z} \rightarrow G$ be a homomorphism such that $\phi(1) = a$. What is $\phi(k)$ for $k \in \mathbb{Z}$?
- (b) Let $\phi: \mathbb{Z}_n \rightarrow G$ be a homomorphism such that $\phi(1) = a$. What is $\phi(k)$ for $k \in \mathbb{Z}_n = \{0, 1, \dots, n-1\}$? What is a^n ?
- (c) Let $\phi: \mathbb{Z} \times \mathbb{Z} \rightarrow G$ be a homomorphism such that $\phi(1, 0) = a$ and $\phi(0, 1) = b$. What is $\phi(k, l)$ for $(k, l) \in \mathbb{Z} \times \mathbb{Z}$? Show that $ab = ba$.
- (8) Recall that we say a subgroup H of a group G is *normal* if $gH = Hg$ for all $g \in G$ (that is, the left and right cosets of H in G coincide). Equivalently, $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. Which of the following subgroups are normal?
- (a) G is an abelian group and H is any subgroup of G .
- (b) $G = S_n$ and $H = A_n$.
- (c) $G = D_n$ and H the subgroup of rotations (including the identity).
- (d) G is a finite group and H is a subgroup of G such that $|H| = \frac{1}{2}|G|$.
- (e) $G = D_n$, $n \geq 3$, and $H = \langle \mu \rangle$ the subgroup generated by a reflection μ . [Hint: If $\rho \in D_n$ is the rotation about the center of the n -gon through angle $2\pi/n$ counter-clockwise, then

$$D_n = \{e, \rho, \rho^2, \dots, \rho^{n-1}, \mu, \rho\mu, \rho^2\mu, \dots, \rho^{n-1}\mu\}$$

and the multiplication table can be determined using the relations $\rho^n = e$, $\mu^2 = e$, and $\mu\rho = \rho^{-1}\mu$.]

- (f) G is any group, and H is the kernel of a homomorphism $\phi: G \rightarrow G'$.
- (g) $G = S_n$, $n \geq 3$, and $H = \{\sigma \in S_n \mid \sigma(n) = n\}$.
- (h) $G = S_4$ and $H = \{e, (12)(34), (13)(24), (14)(23)\}$.

- (9) Suppose H is a normal subgroup of S_n . Let σ be an element of H and σ' a permutation in S_n with the same cycle type as σ (that is, the lengths of the cycles in the expression of σ and σ' as products of disjoint cycles are the same). Show that $\sigma' \in H$. [Hint: H is a normal subgroup of S_n iff $ghg^{-1} \in H$ for all $h \in H$ and $g \in S_n$. How are the cycle decompositions of h and ghg^{-1} related?]
- (10) Let G be a group and $g \in G$ an element. Define a function $i_g: G \rightarrow G$ by $i_g(x) = gxg^{-1}$. [i_g is called “conjugation by g ”.]
- (a) Show carefully that i_g is an isomorphism.
 - (b) Deduce that the order of gxg^{-1} equals the order of x for all $g, x \in G$. [We have seen this before.]
 - (c) Now let H be a subgroup of G , and consider the subset

$$gHg^{-1} := \{ghg^{-1} \mid h \in H\}$$

of G . Show that gHg^{-1} is a subgroup of G which is isomorphic to H . [In particular, $|gHg^{-1}| = |H|$.]

- (11) Let $G = S_3$ and $H = \langle (12) \rangle = \{e, (12)\}$ the cyclic subgroup generated by (12) .
- (a) Compute the left cosets of H in G .
 - (b) Give an example to show that the product of two left cosets of H in G is not always a left coset. [So in this example we cannot define the structure of a group on the set of left cosets. This is because H is not a normal subgroup of G .]
- (12) In each of the following cases, describe the quotient group as simply as possible.
- (a) $\mathbb{Z}/\langle n \rangle$.
 - (b) $\mathbb{Z}_{12}/\langle 3 \rangle$.
 - (c) $\mathbb{Z}_{30}/\langle 22 \rangle$.
 - (d) G/G .
 - (e) $G/\{e\}$.

- (f) $G_1 \times G_2/H_1 \times H_2$ (where H_1 is a normal subgroup of G_1 and H_2 is a normal subgroup of G_2).
 - (g) $G_1 \times G_2/\{e\} \times G_2$.
 - (h) $\mathbb{Z} \times \mathbb{Z}/\langle(0, 5)\rangle$.
 - (i) $U/\langle i \rangle$.
- (13) In each of the following cases, determine the standard abelian group (as in the fundamental theorem) that is isomorphic to G .
- (a) $G = \mathbb{Z}_2 \times \mathbb{Z}_4/\langle(1, 2)\rangle$.
 - (b) $G = \mathbb{Z}_9 \times \mathbb{Z}_9/\langle(3, 3)\rangle$.
 - (c) $G = \mathbb{Z} \times \mathbb{Z}_6/\langle(1, 2)\rangle$.
- (14) Show that $G = D_6$ has a subgroup H isomorphic to D_3 . Show that H is normal and describe the quotient G/H . [Hint: What is the order of D_n ?]. [Note: the same argument shows that D_{2n} has a normal subgroup isomorphic to D_n .]
- (15) Let G be a group and H a normal subgroup of G .
- (a) Show that if G is abelian then G/H is abelian. Give an example where G/H is abelian but G is not abelian.
 - (b) Show that if G is cyclic then G/H is cyclic. Give an example where G/H is cyclic but G is not cyclic.
- (16) Let $G = D_4$ be the group of symmetries of a square. So there is an action of G on the set

$$X = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\} \subset \mathbb{R}^2$$

- (a) Describe all the elements of G geometrically.
- (b) Compute the orbits and stabilizers of the following points of X .
 - i. $(1, 0)$
 - ii. $(1, 1)$
 - iii. $(1/2, 1/2)$
 - iv. $(1/2, 1/3)$

- (17) Let $G = D_4$ be the group of symmetries of the square. Let $A = \{l_1, l_2\}$ be the set of diagonals of the square (that is, the line segments joining opposite vertices). Then G acts on A , or, equivalently, we have a homomorphism $\phi: G \rightarrow S_A$ from G to the group of permutations of A . Determine the kernel and image of ϕ , and identify them with standard groups.
- (18) Use the orbit-stabilizer theorem to determine the order of the symmetry group of a cube.
- (19) A *regular tetrahedron* is a 3-dimensional object with 4 faces, each of which is an equilateral triangle. Find the order of the symmetry group of a regular tetrahedron.
- (20) Consider the groups $G_1 = \mathbb{Z}_{12}$, $G_2 = S_{12}$, $G_3 = \mathbb{Z}_2 \times \mathbb{Z}_6$, $G_4 = D_6$, $G_5 = A_4$. Show that no two of these groups are isomorphic. [Hint: To distinguish D_6 and A_4 , count the number of elements of some particular order.]