Math 300.2 Midterm 1 review questions

Paul Hacking

February 27, 2012

Reading: Gilbert and Vanstone, Chapters 1,2,4.

(1) Give the truth tables for the following statements.

(a) $P \Rightarrow Q$.

(b) $\mathrm{NOT}(P \,\mathrm{OR}\, Q)$.

(c) $(P \,\mathrm{AND}\, Q) \Rightarrow R$.

(2) Show that the following statements are equivalent

(a) $\mathrm{NOT}(P \,\mathrm{AND}\, Q)$, $(\mathrm{NOT}\, P) \,\mathrm{OR} (\mathrm{NOT}\, Q)$.

(b) $P \Rightarrow (Q \,\mathrm{OR}\, R)$, $(P \,\mathrm{AND} (\mathrm{NOT}\, Q)) \Rightarrow R$.

(3) Describe the following sets explicitly. [Recall: $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of integers, and $\mathbb{R}$ is the set of real numbers.]

(a) $\{x \in \mathbb{Z} \mid 25 < x^2 < 50\}$.

(b) $\{x \in \mathbb{R} \mid x^2 = x + 1\}$.

(c) $\{x \in \mathbb{N} \mid x \text{ divides } 30\}$.

(4) Recall if A and B are sets then

$$A \cup B = \{x \mid (x \in A) \,\mathrm{OR} (x \in B)\}$$

$$A \cap B = \{x \mid (x \in A) \,\mathrm{AND} (x \in B)\}$$

$$A \setminus B = \{x \mid (x \in A) \,\mathrm{AND} (x \notin B)\}$$

Show that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

using (a) a Venn diagram (b) a truth table.

(5) Translate the following mathematical statements into an english sentence.

(a) $\forall x \in \mathbb{R} \quad e^x > 0$.

(b) $\exists x \in \mathbb{R} \quad x^3 = 7$.

(c) $\exists b \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad |\sin(x)| \leq b$.

(d) $\forall a, b \in \mathbb{N} \quad ((\gcd(a,b) = 1) \Rightarrow (\exists x, y \in \mathbb{Z} \quad ax + by = 1))$.

(6) Negate the following statements.

(a) $\exists x \in \mathbb{R} \quad \forall y \in \mathbb{R} \quad y \leq x$.

(b) $\forall a \in \mathbb{N} \quad \exists b, c \in \mathbb{Z} \quad (a = b^2 + c^2)$.

(7) Prove the following statements for each $n \in \mathbb{N}$ by induction.

(a) $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

(b) $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$.

(c) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

(8) (a) Prove that $\binom{2n}{n} \geq 2^n$ for $n \in \mathbb{N}$. [Hint: Use the binomial coefficient formula $\binom{m}{r} = \frac{m!}{r!(m-r)!}$ and induction.]

(b) The Fibonnaci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove that $F_n \leq 2^n$ for $n \in \mathbb{N}$ using strong induction.

(c) Guess a formula for the sum

$$f(n) = 1 - 3 + 5 - \cdots + (-1)^{n-1}(2n-1) = \sum_{i=1}^{n} (-1)^{i-1}(2i-1).$$

Prove your formula is correct using induction.

(9) (a) Let $x \in \mathbb{R}$, $x \geq 0$. Prove that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$ using the binomial theorem.

(b) Let $x \in \mathbb{R}$, $x \geq -1$. Prove that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$ using induction.

(10) Prove the following identity for each $n \in \mathbb{N}$:

$$(1-x^{(2^n)}) = (1-x^2)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{(2^{n-1})}) = (1-x^2)\prod_{i=1}^{n-1}(1+x^{(2^i)}).$$

[Hint: Proof by induction. Use the "difference of two squares" identity $a^2 - b^2 = (a+b)(a-b)$ and the law of exponents $x^{pq} = (x^p)^q$.]

(11) The downtown portion of a city is a rectangular grid. A pedestrian must travel from one street corner to another, a total distance of 5 blocks north and 3 blocks east. How many routes are there which are as short as possible? How many routes are there if the pedestrian must travel m blocks north and n blocks east, for some $m, n \in \mathbb{N}$? [Hint: The answer is a binomial coefficient]

(12) Find all solutions $x, y \in \mathbb{Z}$ to the following equations or prove that there are none. Explain your work carefully.

(a) $15x + 57y = 4$
(b) $13x + 30y = 2$.
(c) $14x = 35y + 21$.

(13) Compute the greatest common divisor of the following pairs of integers.

(a) 24 and 45.
(b) $2^2 3^3 5^2 7^4$ and $2^5 3^4 11^6$.
(c) 343 and 667

(14) For each of the following statements, prove or give a counter example.

(a) If $a \mid b$ and $b \mid c$ then $a \mid c$.
(b) If $a \mid b$ and $a \mid c$ then $a \mid 3b - 5c$.
(c) If $a \mid bc$ then $a \mid b$ or $a \mid c$.

(15) Let p be a prime number and $a \in \mathbb{Z}$. Prove that $\gcd(a, p) = p$ if p divides a and $\gcd(a, p) = 1$ otherwise.

(16) Let $a, b, c \in \mathbb{N}$ be positive integers. Let $d = \gcd(a, b)$ and $e = a/d \in \mathbb{Z}$. Prove that $a \mid bc \iff e \mid c$.