Math 300 Midterm 2, Wednesday 4/11/12, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 7 questions for a total of 80 points. Calculators, notes, and textbooks are not allowed. Justify all your answers carefully. If you use a result proved in the textbook or class notes, state the result precisely.

Q1 (12 points). Find all solutions to the following congruences.

(a) (4 points) $x^2 - x + 3 \equiv 0 \bmod 5$.

(b) (4 points) $13x \equiv 5 \bmod 29$

(c) (4 points) $x \equiv 3 \bmod 5$ and $x \equiv 1 \bmod 6$.

Q2 (8 points). Let n be a positive integer with digits $a_r, a_{r-1}, \ldots, a_0$ (in the usual base 10 notation). Show that

$$n \equiv a_r + a_{r-1} + \cdots + a_0 \bmod 9.$$

Q3 (13 points). Let R be a relation on a set S. What does it mean to say that R is an equivalence relation?(3 points). For each of the following relations R, either prove carefully that R is an equivalence relation or show by example that one of the required properties is not satisfied.

(a) (3 points) $S = \mathbb{R}$, $xRy \iff x - y = 2\pi k$ for some $k \in \mathbb{Z}$.

(b) (3 points) $S = \mathbb{R}$, $xRy \iff |x - y| < 3$.

(c) (4 points) $S = \mathbb{N}$, $aRb \iff ab$ is a square, that is, $ab = n^2$ for some $n \in \mathbb{N}$.

Q4 (12 points). For each of the following functions, compute the inverse or prove that no inverse exists.

(a) (3 points) $f\colon \{1, 2, 3, 4\} \to \{A, B, C, D\}$, $f(1) = D$, $f(2) = B$, $f(3) = A$, $f(4) = C$.

(b) (3 points) $f\colon [0, \pi] \to [0, 1]$, $f(x) = \sin(x)$.

(c) (3 points) $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = x^2 + 4$

(d) (3 points) $f\colon [0, \infty) \to (0, 1]$, $f(x) = 1/(1 + x^2)$.

Q5. (11 points) Let X, Y, Z be sets and $f\colon X \to Y$, $g\colon Y \to Z$ be functions. Define the composite function $g \circ f$ (2 points).

(a) (3 points) Show that if f is injective and g is injective then $g \circ f$ is injective.

(b) (3 points) Show that if $g \circ f$ is surjective then g is surjective.

(c) (3 points) Give an example of two functions $f\colon X \to Y$ and $g\colon Y \to Z$ such that $g \circ f$ is surjective but f is not surjective.

Q6 (12 points). For each of the following pairs of sets X and Y, describe a bijection $f\colon X \to Y$ or prove that no such bijection exists.

(a) (4 points) $X = \mathbb{N}$, $Y = \{n \in \mathbb{N} \mid n \geq 11\}$.

(b) (4 points) $X = [2, 4]$, $Y = [5, 8]$.

(c) (4 points) $X = \mathbb{Q}$, $Y = \mathbb{R}$.

Q7 (12 points). What does it mean to say that a set X is countable? (2 points). Let X be a countable set and Y a finite set.

(a) (5 points) Show that $X \cup Y$ is countable.

(b) (5 points) Show that $X \times Y$ is countable.