Math 300 Midterm 1, Wednesday 2/29/12, 7PM-8:30PM.

Instructions: Exam time is 90 mins. There are 8 questions for a total of 70 points. Calculators, notes, and textbooks are not allowed. Justify all your answers carefully. If you use a result proved in the textbook or class notes, state the result precisely.

Q1. (8 points) For each of the following pairs of statements, compute the truth tables and use them to determine whether the statements are equivalent.

(a) (4 points) $\mathrm{NOT}(P \Rightarrow Q)$, $P\,\mathrm{AND}(\mathrm{NOT}\,Q)$.

(b) (4 points) $(P\,\mathrm{AND}\,Q)\,\mathrm{OR}\,R$, $(P\,\mathrm{OR}\,R)\,\mathrm{AND}(Q\,\mathrm{OR}\,R)$.

Q2 (8 points). A sequence $a_1, a_2, a_3, \ldots$ of integers is defined by $a_1 = 1$ and $a_{n+1} = 2a_n + 5$ for each $n \in \mathbb{N}$. Prove that $a_n = 3 \cdot 2^n - 5$ for all $n \in \mathbb{N}$.

Q3 (8 points). Let $n \in \mathbb{N}$. Show that if $n \geq 3$ then $5 \cdot (n!) \geq 3^n$.

Q4 (11 points). Let a, b, c be integers. What does it mean to say that a divides b ? (2 points). For each of the following statements give a proof or a counterexample. [We use the notation $a \mid b$ for "a divides b".]

(a) (3 points) If $a \mid b$ and $a \mid c$ then $a \mid 4b + 7c$.

(b) (3 points) If $a \mid bc$ then $a \mid b$ or $a \mid c$.

(c) (3 points) If $a \mid b$ and $b \mid c$ then $a \mid c$.

Q5 (10 points) Let a, b be integers, not both zero. Give the definition of the greatest common divisor $\gcd(a, b)$ of a and b (2 points). Compute the greatest common divisor of the following pairs of integers.

(a) (2 points) 12 and 18.

(b) (3 points) $2^5 \cdot 3^4 \cdot 5^6 \cdot 11^2$ and $2^2 \cdot 5 \cdot 7^3 \cdot 13^6$.

(c) (3 points) 217 and 103.

Q6 (8 points) For each of the equations below, find all solutions $x, y \in \mathbb{Z}$ or explain why no solutions exist.

(a) (4 points) $5x + 13y = 6$.

(b) (4 points) $42x + 57y = 107$.

Q7 (8 points). Let n be an integer such that $n > 1$. What does it mean to say that n is prime ? (2 points). Now let a and b be positive integers. Show that the following statements are equivalent (6 points):

(i) $\gcd(a, b) \neq 1$.

(ii) There exists a prime p such that $p \mid a$ and $p \mid b$.

Q8 (9 points). Let $n, r \in \mathbb{Z}$ be integers such that $0 \leq r \leq n$. Give the definition of the number $\binom{n}{r}$ (pronounced "n choose r") (2 points).

(a) (4 points) Show that if p is prime and $1 \leq r \leq p-1$ then p divides $\binom{p}{r}$.

(b) (3 points) Using part (a) or otherwise, show that if p is prime and $a, b \in \mathbb{Z}$ then p divides $(a+b)^p - a^p - b^p$.