Math 300.2 Final exam review questions

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Reading: Gilbert and Vanstone, Chapters 1,2,3,4,5,6,8.

(1) Let P and Q be statements.

(a) What is the contrapositive of $P \Rightarrow Q$? Use a truth table to show that $P \Rightarrow Q$ and its contrapositive are equivalent.

(b) What is the converse of the statement $P \Rightarrow Q$? Use a truth table to show that $P \Rightarrow Q$ and its converse are *not* equivalent.

(2) Let A, B, C be sets.

(a) Define the union $A \cup B$, the intersection $A \cap B$, and the difference $A \setminus B$.

(b) Show using a truth table that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Check this result using a Venn diagram.

(c) Show using a truth table that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$. Check this result using a Venn diagram.

(3) Translate the following statements into english sentences.

(a) $\forall x \in \mathbb{N} \quad x \geq 1$

(b) $\forall x \in \mathbb{R} \quad x^2 \geq 0$

(c) $\exists x \in \mathbb{R} \quad x^2 - 6x + 7 = 0$

(d) $\exists x \in \mathbb{Z} \quad x^2 \equiv 2 \bmod 7$

(e) $\forall z, w \in \mathbb{C} \quad (zw = 0 \Rightarrow ((z = 0)\,\text{OR}(w = 0)))$

(f) $\forall x \in \mathbb{N} \quad \exists y \in \mathbb{N} \quad y > x$

(g) $\forall y \in \mathbb{R} \quad \exists x \in \mathbb{R} \quad x^3 = y$

(4) Negate the following statements, then translate into an english sentence.

(a) $\exists x \in \mathbb{Z} \quad x^2 \equiv 3 \bmod 4$

(b) $\forall x \in \mathbb{R} \quad x^2 - 4x + 2 > 0$

(c) $\forall x \in \mathbb{N} \quad \exists y \in \mathbb{N} \quad y < x$

(d) $\exists b \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad \log x \leq b$

(e) $\exists x, y, z \in \mathbb{N} \quad x^3 + y^3 = z^3$

(5) Translate the following sentences into mathematical statements using quantifiers.

(a) $x^2 + 2x + 3$ is positive for all real numbers x.

(b) There is a real number x such that $x^2 = 2$.

(c) For every positive integer n there is a real number a such that $e^x \geq x^n$ for $x \geq a$.

(d) There is a real number b such that $x - x^2 \leq b$ for all real numbers x.

(6) Define a sequence of integers $a_1, a_2, a_3, \ldots$ recursively by $a_1 = 10$ and $a_{n+1} = 3a_n - 8$ for $n \in \mathbb{N}$. Prove that $a_n = 2 \cdot 3^n + 4$ for all $n \in \mathbb{N}$.

(7) Prove that

$$\sum_{r=1}^{n} (2r+1) = 3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$$

for each $n \in \mathbb{N}$.

(8) Prove that

$$\sum_{r=1}^{n} r(r+2) = 1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{1}{6} n(n+1)(2n+7)$$

for each $n \in \mathbb{N}$.

(9) Let $p(x)$ be a polynomial of degree n with real coefficients. That is,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Prove by induction on n that the equation $p(x) = 0$ has at most n real solutions.

(10) Prove that $5^n > 4^n + 3^n + 2^n$ for $n \geq 3$.

(11) Prove that $5^{2n} - 3^n \equiv 0 \bmod 11$ for all $n \in \mathbb{N}$.

(12) Define the Fibonacci numbers F_n for $n \in \mathbb{N}$ by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

- (a) Write down the first few terms of the Fibonnaci sequence $F_1, F_2, F_3, \ldots$.
- (b) Prove that $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$ for each $n \in \mathbb{N}$.

(13) Let $a, b \in \mathbb{N}$. Define the greatest common divisor $\gcd(a, b)$. Compute the greatest common divisor of the following pairs of integers

- (a) $123, 39$.
- (b) $157, 83$.
- (c) $2^5 \cdot 3^7 \cdot 5^9 \cdot 11^4$, $2 \cdot 3^2 \cdot 7^{10}$.

(14) Find all solutions $x, y \in \mathbb{Z}$ of the following equations.

- (a) $24x + 52y = 8$
- (b) $42x + 15y = 7$

(15) What does it mean to say a positive integer $n > 1$ is prime?

- (a) State the fundamental theorem of arithmetic.
- (b) List all the positive divisors of 72.
- (c) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factorization of a positive integer n. How many positive divisors of n are there?

(16) Find all solutions of the following congruences.

- (a) $x \equiv 2 \bmod 7$ and $x \equiv 3 \bmod 11$.
- (b) $5x \equiv 12 \bmod 17$.

(c) $x^2 + 3x + 1 \equiv 0 \bmod 5$.

(17) Prove that congruence modulo m defines an equivalence relation on the set $\mathbb{Z}$.

(18) State Fermat's little theorem. Let p be a prime and let $\alpha \in \mathbb{N}$ be such that $\gcd(\alpha, p-1) = 1$

(a) Explain why there exists $\beta \in \mathbb{N}$ such that $\alpha\beta \equiv 1 \bmod (p-1)$

(b) Let $X = \{1, 2 \ldots, p-1\}$. Show that the function $f \colon X \to X$, $f(x) = x^\alpha \bmod p$ has inverse $g \colon X \to X$ given by $g(x) = x^\beta \bmod p$, where β is the number from part (a).

(19) Let $m \in \mathbb{N}$ and let $a \in \mathbb{N}$ be such that $\gcd(a, m) = 1$

(a) Let $X = \{0, 1, 2, \ldots, m-1\}$. Show that the function $f \colon X \to X$ given by $f(x) = ax \bmod m$ is injective.

(b) Deduce that f is bijective.

(20) Let S be a set and R a relation on S. What does it mean to say that R is an equivalence relation? In each of the following cases, determine whether R is an equivalence relation.

(a) $S = \mathbb{Z}$, $aRb \iff a \leq b$.

(b) $S = \mathbb{Q}$, $aRb \iff b = a \cdot 2^n$ for some $n \in \mathbb{Z}$.

(c) $S = \mathbb{C}$, $zRw \iff |z - w| \leq 1$

(d) S is the set of all lines in the plane, and for two lines l and m we define $lRm \iff l$ is parallel to m.

(21) Let $S = \mathbb{R}^2 \setminus \{(0,0)\}$ and R be the relation on S defined by

$$(x_1, y_1)R(x_2, y_2) \iff (x_2, y_2) = \lambda(x_1, y_1) \text{ for some positive real number } \lambda.$$

(a) Show that R is an equivalence relation.

(b) Draw a picture showing the equivalence classes of R.

(c) Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the circle with center the origin and radius 1. Let f be the function

$$f \colon C \to S/R$$

from the circle C to the set S/R of equivalence classes of R given by $f(x,y) = [(x,y)]$ (that is, $f(x,y)$ is the equivalence class of (x,y)). Show that f is a bijection.

(22) Let X and Y be sets and $f\colon X \to Y$ a function from X to Y. What does it mean to say that f is injective? What does it mean to say that f is surjective. For each of the following functions determine whether f is injective and whether f is surjective. (Justify your answers carefully.)

- (a) $f\colon \mathbb{N}^2 \to \mathbb{N}$, $f(a,b) = 3^a \cdot 7^b$.
- (b) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x$.
- (c) $f\colon [0,\pi] \to \mathbb{R}$, $f(x) = 2\sin x + 5$.
- (d) $f\colon \mathbb{C} \to \mathbb{R}$, $f(z) = |z|$.

(23) Let X and Y be sets and $f\colon X \to Y$ a function from X to Y. What condition must f satisfy in order to have an inverse? In each of the following cases determine whether f has an inverse and if so describe the inverse explicitly.

- (a) $f\colon (0,\infty) \to \mathbb{R}$, $f(x) = 3\log_e x + 5$.
- (b) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = 4x + 7$.
- (c) $f\colon [0,1] \to [7,11]$, $f(x) = x^2 + 3x + 7$.

(24) Let $a, b \in \mathbb{N}$.

- (a) Show that the function $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ defined by $f(x,y) = ax + by$ is surjective if and only if $\gcd(a,b) = 1$.
- (b) Is f injective? Justify your answer.

(25) Let X, Y be sets and $f\colon X \to Y$, $g\colon Y \to X$ be functions.

- (a) Show that if $g(f(x)) = x$ for all $x \in X$ and f is surjective, then $f(g(y)) = y$ for all $y \in Y$. (So g is the inverse of f and f and g are bijections).
- (b) Give an example of functions f and g such that $g(f(x)) = x$ for all $x \in X$ but $f(g(y)) \neq y$ for some $y \in Y$.

(26) Let X be a set. What does it mean to say that X is countable? In each of the following cases, determine whether the set is countable. Justify your answer carefully.

(a) $X = \{n \in \mathbb{Z} \mid n \equiv 3 \bmod 5\}$.

(b) $X = \{x \in \mathbb{R} \mid (x > 0)\,\text{AND}(x^2 \in \mathbb{Q})\}$.

(c) X the set of subsets of $\mathbb{N}$ of size 2, that is

$$X = \{\{a, b\} \mid a, b \in \mathbb{N}, a < b\}.$$

[Hint: use the fact that $\mathbb{N} \times \mathbb{N}$ is countable.]

(d) $X = \mathcal{P}(\mathbb{Q})$, the power set of $\mathbb{Q}$. [Recall that the power set $\mathcal{P}(Y)$ of a set Y is the set of all subsets of Y.]

(e) X the set of functions $f : \mathbb{N} \to \{0, 1\}$ from the set of positive integers to the set $\{0, 1\}$. [Hint: Modify Cantor's diagonal argument.]

(27) Recall that in class we showed that $\mathbb{Q}$ is countable, that is, there is a bijection $f : \mathbb{N} \to \mathbb{Q}$. Show that there does *not* exist a bijection $g : \mathbb{N} \to \mathbb{Q}$ such that $g(a) < g(b)$ when $a < b$.

(28) Let X be the set of all finite subsets of $\mathbb{N}$. In class we showed that X is countable. Here we will describe another proof of this fact. Define a function $f : X \to \mathbb{N}$ as follows. Let $p_1, p_2, p_3, \ldots = 2, 3, 5, \ldots$ be the list of prime numbers in increasing order. For $S \in X$, write $S = \{a_1, a_2, \ldots, a_r\}$ where $a_1 < a_2 < \ldots < a_r$ and $r = |S|$. We define $f(S) = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$.

(a) Explain why f is injective.

(b) Deduce that X is countable.

(c) Is f surjective?

(29) Find all solutions $z \in \mathbb{C}$ of the following equations.

(a) $z^2 - 6z + 25 = 0$

(b) $z^3 = -8i$.

(c) $z^4 - 2z^3 + 6z^2 - 2z + 5 = 0$, given that $z = i$ is a solution.