1. $f(x) = |x| + |x-1| = \begin{cases} x + x-1 & \text{if } x \geq 1 \\ x-(x-1)=1 & \text{if } 0 \leq x \leq 1 \\ -x-(x-1) & \text{if } x \leq 0 \end{cases}$

and $x + x - 1 = 2x-1 \geq 2\cdot 1 - 1 = 1$ if $x \geq 1$

$-x-(x-1) = -2x+1 \geq -2\cdot 0 + 1 = 1$ if $x \leq 0$.

2. a. cases:

x mod 6	x^2+x mod 6
0	$0^2+0 = 0$
1	$1^2+1 = 2$
2	$2^2+2 = 6 \equiv 0$ mod 6
3	$3^2+3 = 12 \equiv 0$ mod 6
4	$4^2+4 = 20 \equiv 2$ mod 6
5	$5^2+5 = 30 \equiv 0$ mod 6.

So, $x^2+x \equiv 0 \mod 6 \iff x \equiv 0, 2, 3,$ or $5 \mod 6$.

b. cases:

x mod 7	x^3+1 mod 7
0	$0^3+1 = 1$
1	$1^3+1 = 2$
2	$2^3+1 = 9 \equiv 2$ mod 7
3	$3^3+1 = 28 \equiv 0$ mod 7
4	$4^3+1 = 65 \equiv 2$ mod 7
5	$5^3+1 = 126 \equiv 0$ mod 7
6	$6^3+1 = 217 \equiv 0$ mod 7

So, $x^3+1 \equiv 0 \mod 7 \iff x \equiv 3, 5,$ or $6 \mod 7$

(<u>Note</u>: Can make calculations faster as follows:

$$5^3+1 = 5\cdot 5^2+1 = 5\cdot 25+1 \equiv 5\cdot 4+1 = 21 \equiv 0 \mod 7)$$

↑ because $25 \equiv 4 \mod 7$

3 a. Cases

$x \bmod 5$	0	1	2	3	4
$x^2 \bmod 5$	0	1	4	$9 \equiv 4$	$16 \equiv 1$

So $x^2 \equiv 0, 1$ or $4 \bmod 5$.

In particular, $x^2 \equiv 2 \bmod 5$ has no solutions.

b. __Claim:__ $x^2 - 5y^2 = 7$ has no solutions $x, y \in \mathbb{Z}$.

__Proof:__ By contradiction.

Suppose $\exists\, x, y \in \mathbb{Z}$ such that $x^2 - 5y^2 = 7$

Then $x^2 - 5y^2 \equiv 7 \bmod 5$

Now $x^2 - 5y^2 \equiv x^2 - 0 \cdot y^2 \equiv x^2 \bmod 5$

and $7 \equiv 2 \bmod 5$

So $x^2 \equiv 2 \bmod 5$ ※ this contradicts part (a). □.

4. __Claim:__ $\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$

__Proof:__ By induction.

__$n=1$__ : LHS $= 1^2 = 1$, RHS $= \frac{1}{6}\, 1 \cdot (1+1)(2 \cdot 1 + 1) = 1$ ✓

__$n=k \Rightarrow n=k+1$__ : Suppose $\sum_{r=1}^{k} r^2 = \frac{1}{6} k(k+1)(2k+1)$ (※)

We will show $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$:-

LHS $= \sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$

$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$ by inductive hypothesis (※)

$= \frac{1}{6}(k+1)(k(2k+1) + 6(k+1)) = \frac{1}{6}(k+1)(2k^2 + 7k + 6)$

RHS $= \frac{1}{6}(k+1)\underbrace{(k+2)(2k+3)}_{\text{expand}} = \frac{1}{6}(k+1)(2k^2 + 7k + 6)$ ✓ □.

5 a. $f(x) = (x-\alpha)\cdot g(x) + r$

$\Rightarrow f(\alpha) = (\alpha-\alpha)\cdot g(\alpha) + r$

$= 0 \cdot g(\alpha) + r = r$

b. <u>Claim</u> If f is a polynomial of degree n then there are at most n real solutions of the equation $f(x) = 0$.

<u>Proof</u>: By induction on n.

<u>$n=1$</u>: $f(x) = a_1 x + a_0$, $a_0, a_1 \in \mathbb{R}$, $a_1 \neq 0$.

$f(x) = 0 \iff a_1 x + a_0 = 0 \iff x = -a_0/a_1$

So, the equation $f(x) = 0$ has one real solution.

<u>$n=k \Rightarrow n=k+1$</u>: Let f be a polynomial of degree $k+1$.

If the equation $f(x) = 0$ has no real solutions, we are done.

Otherwise, let $\alpha \in \mathbb{R}$ be a solution.

Using part (a), we have $f(x) = (x-\alpha)\cdot g(x)$,

for some polynomial $g(x)$ of degree $(k+1)-1 = k$.

By the inductive hypothesis, the equation $g(x) = 0$ has at most k solutions.

And $f(x) = (x-\alpha)\cdot g(x) = 0 \iff (x-\alpha) = 0$ OR $g(x) = 0$

$\iff x = \alpha$ OR $g(x) = 0$.

So $f(x) = 0$ has at most $1 + k = k+1$ real solutions. □.

6 <u>Claim</u> For all $n \in \mathbb{N}$ such that $n \geq 5$, $n! \geq 3^{n-1}$

<u>Proof</u> By induction.

<u>$n=5$</u>: LHS $= 5! = 120$

RHS $= 3^4 = 81$. $120 \geq 81$ ✓

<u>$n=k \Rightarrow n=k+1$</u>: Suppose $k \geq 5$ and $k! \geq 3^{k-1}$

We will show $(k+1)! \geq 3^{(k+1)-1} = 3^k$.

$$(k+1)! = (k+1)\cdot k! \geqslant (k+1)\cdot 3^{k-1} \geqslant (5+1)\cdot 3^{k-1} = 6\cdot 3^{k-1}$$

$$\geqslant 3\cdot 3^{k-1}$$

$$= 3^k. \quad \square$$

(↑ inductive hypothesis $k! \geqslant 3^{k-1}$; ↑ $k \geqslant 5$)

7. **Claim** $\forall n \in \mathbb{N},\ a_n = 3\cdot 2^n - 5$

Proof By strong induction.

For $k \in \mathbb{N}$, we assume $a_m = 3\cdot 2^m - 5$ for all $m \in \mathbb{N}$ such that $m < k$, and show $a_k = 3\cdot 2^k - 5$.

If $k=1$, $a_1 = 1$ and $3\cdot 2^1 - 5 = 1$ ✓

If $k=2$, $a_2 = 7$ and $3\cdot 2^2 - 5 = 12 - 5 = 7$ ✓

If $k \geqslant 3$, $a_k = 3a_{k-1} - 2a_{k-2}$

$$= 3(3\cdot 2^{k-1} - 5) - 2(3\cdot 2^{k-2} - 5) \quad \text{by the inductive hypothesis}$$

$$= 9\cdot 2^{k-1} - 15 - 6\cdot 2^{k-2} + 10$$

$$= (9\cdot 2 - 6)\, 2^{k-2} - 5 \quad \text{writing } 2^{k-1} = 2\cdot 2^{k-2}$$

$$= 12\cdot 2^{k-2} - 5$$

and $3\cdot 2^k - 5 = (3\cdot 2^2)\cdot 2^{k-2} - 5 = 12\cdot 2^{k-2} - 5$ ✓ $\square$

8. For $a, b, c \in \mathbb{Z}$ such that a & b are not both zero, the equation $ax + by = c$ has a solution $x, y \in \mathbb{Z}$ if and only if $\gcd(a,b) \mid c$

Our case: $492x + 213y = c \quad (*)$

Compute $\gcd(492, 213)$ by Euclidean algorithm

$492 = 2\cdot 213 + 66$

$213 = 3\cdot 66 + 15$ So $\gcd(492, 213) = 3$.

$66 = 4\cdot 15 + 6$

$15 = 2\cdot 6 + \boxed{3}$, $6 = 2\cdot 3 + 0$.

Therefore, the equation $(*)$ has a solution $x,y\in\mathbb{Z}$ $\iff$ $3|c$.

9. $52x+91y=26$ $(\dagger)$

Compute $\gcd(52,91)$:-

1 $91 = 1\cdot 52 + 39$

2 $52 = 1\cdot 39 + \boxed{13}$ $\gcd(91,52)=13$.

$39 = 3\cdot 13 + 0$

Note that $13 \mid 26$. So equation $(\dagger)$ has a solution $x,y\in\mathbb{Z}$.

We can find one solution by "back substitution" in the Euclidean algorithm:-

$$13 \overset{2}{=} 52-1\cdot 39 \overset{1}{=} 52-1\cdot(91-1\cdot 52) = 2\cdot 52 - 1\cdot 91$$

So $52u+91v=13$ has one solution $u=2, v=-1$

$\Rightarrow$ $52x+91y=26=2\cdot 13$ has one solution $x=2u=4$, $y=2v=-2$.

For $a,b,c\in\mathbb{Z}$, a,b not both zero, if $x_0,y_0 \in\mathbb{Z}$ is one solution of

$ax+by=c$, then all solutions $x,y\in\mathbb{Z}$ are given by

$$\left.\begin{aligned} x &= x_0 + \tfrac{b}{d}\cdot k \\ y &= y_0 - \tfrac{a}{d}\cdot k \end{aligned}\right\} \text{ where } d=\gcd(a,b) \text{ and } k\in\mathbb{Z} \text{ is arbitrary.}$$

In our case $52x+91y=26$, $x_0=4$, $y_0=-2$, $d=\gcd(52,91)=13$

$$\left.\begin{aligned} \text{so } x &= 4+\tfrac{91}{13}\cdot k = 4+7k \\ y &= -2-\tfrac{52}{13}k = -2-4k \end{aligned}\right\} \text{ where } k\in\mathbb{Z} \text{ is arbitrary}$$

10. $42x\equiv 12 \bmod 57$ $\iff$ $42x=12+57q$, some $q\in\mathbb{Z}$

$\iff$ $42x+57y=12$, some $y=-q\in\mathbb{Z}$.

$\gcd(42,57) = ?$

1) $57 = 1\cdot 42 + 15$

2) $42 = 2\cdot 15 + 12$

3) $15 = 1\cdot 12 + \boxed{3}$ $\qquad \gcd(42,57) = 3$. Check $3 \mid 12$ ✓

$12 = 4\cdot 3 + 0$ $\qquad 12 = 4\cdot 3$

$3 \overset{3)}{=} 15 - 1\cdot 12 \overset{2)}{=} 15 - 1\cdot(42 - 2\cdot 15) = 3\cdot 15 - 1\cdot 42$

$\overset{1)}{=} 3\cdot(57 - 1\cdot 42) - 1\cdot 42 = 3\cdot 57 - 4\cdot 42$

So $u = -4$, $v = 3$ is a solution of $42u + 57v = 3$

$\Rightarrow x = 4u = -16$, $y = 4v = 12$ is a solution of $42x + 57y = 12 = 4\cdot 3$

All solutions: $x = -16 + \frac{57}{3}\cdot k = -16 + 19k$, $y = 12 - \frac{42}{3}\cdot k = 12 - 14k$, for $k \in \mathbb{Z}$ arbitrary

So, all solutions of $42x \equiv 12 \bmod 57$ are given by

$x = -16 + 19k$, $k \in \mathbb{Z}$ arbitrary,

i.e. $x \equiv -16 \equiv 3 \bmod 19$.

11. __Claim__ There does __not__ exist $x \in \mathbb{Z}$ such that $x \equiv 9 \bmod 14$ and $x \equiv 5 \bmod 21$.

__Proof__ By contradiction.

Suppose there __does__ exist $x \in \mathbb{Z}$ such that $x \equiv 9 \bmod 14$ and $x \equiv 5 \bmod 21$.

So $x = 9 + 14q$, some $q \in \mathbb{Z}$

and $x = 5 + 21r$, some $r \in \mathbb{Z}$.

Then $9 + 14q = 5 + 21r$

$14q - 21r = -4$

But $\gcd(14,-21) = \gcd(14,21) = 7 \nmid -4$. # □.

12. a. If $d \in \mathbb{N}$ and $d \mid n$ and $d \mid n+7$,
then $d \mid (n+7) - n = 7$.
So $d = 1$ or 7
$\therefore$ $\gcd(n, n+7) = 1$ or 7 (& $\gcd = 7 \iff 7 \mid n$)

b. $\gcd(7n+11, 3n+5) = ?$
Euclidean algorithm:-

$$7n+11 = 2\cdot(3n+5) + (n+1)$$
$$3n+5 = 3\cdot(n+1) + 2$$

So, $\gcd(7n+11, 3n+5) = \gcd(3n+5, n+1) = \gcd(n+1, 2)$

$$= \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

13. a. $a \in \mathbb{N}$, $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $p_1, \ldots, p_r$ distinct primes
$\alpha_1, \ldots, \alpha_r \in \mathbb{Z}_{\geq 0}$.

Then the $d \in \mathbb{N}$ such that $d \mid a$
are given by $d = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_r^{\gamma_r}$ where $0 \leq \gamma_i \leq \alpha_i$ for each i.

It follows that if $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ & $b = p_1^{\beta_1} \cdots p_r^{\beta_r}$
then

$$\gcd(a,b) = p_1^{\min(\alpha_1, \beta_1)} \cdots p_r^{\min(\alpha_r, \beta_r)}$$

where $\min(\alpha, \beta) = \begin{cases} \alpha & \text{if } \alpha \leq \beta \\ \beta & \text{if } \beta \leq \alpha \end{cases}$

Similarly,

$$\operatorname{lcm}(a,b) = p_1^{\max(\alpha_1, \beta_1)} \cdots p_r^{\max(\alpha_r, \beta_r)}$$

where $\max(\alpha, \beta) = \begin{cases} \beta & \text{if } \alpha \leq \beta \\ \alpha & \text{if } \beta \leq \alpha. \end{cases}$

b.

Note $\min(\alpha,\beta) + \max(\alpha,\beta) = \alpha+\beta$

So $\gcd(a,b)\cdot \operatorname{lcm}(a,b) = p_1^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)} \cdots p_r^{\min(\alpha_r,\beta_r)+\max(\alpha_r,\beta_r)}$

$= p_1^{\alpha_1+\beta_1} \cdots p_r^{\alpha_r+\beta_r}$

$= (p_1^{\alpha_1} \cdots p_r^{\alpha_r})(p_1^{\beta_1} \cdots p_r^{\beta_r})$

$= a \cdot b.$ □.

14. Claim: If p, q are consecutive odd primes then $p+q$ has at least 3 prime factors (not necessarily different).

Proof: Recall the fundamental theorem of arithmetic: For all $n \in \mathbb{N}$, we can write $n = p_1 p_2 \cdots p_r$ where $r \in \mathbb{Z}_{\geq 0}$ and $p_1, p_2, \ldots, p_r$ are primes (not necessarily different), and this expression is unique up to reordering the factors.

In our case we have $p+q = p_1 p_2 \cdots p_r$ & we must show $r \geq 3$.

Proof by contradiction.

Suppose $r < 3$.

Note that $2 \mid p+q$ (because p & q are odd)

and $p+q > 2$ (because $p > 1$ & $q > 1$ since p & q are prime).

So $r \geq 2$ ($r \neq 1$ because $p+q$ is not prime),

and $r = 2$ by our assumption.

So $p+q = p_1 p_2$. Also, as already noted, $2 \mid p+q$, so (reordering the factors if necessary) we may assume $p_1 = 2$,

$p+q = 2p_2$

Now $p_2 = \frac{p+q}{2}$, and $p < \frac{p+q}{2} < q$ #

$\because$ p & q are consecutive primes. □.

15 a) $(x \in A \setminus (B \cap C)) \equiv P \text{ AND } (\text{NOT}(Q \text{ AND } R))$

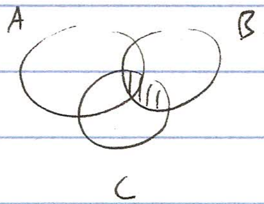
$(x \in (A \setminus B) \cup (A \setminus C)) \equiv (P \text{ AND } (\text{NOT } Q)) \text{ OR } (P \text{ AND } (\text{NOT } R))$

b) $P \text{ AND } (\text{NOT}(Q \text{ AND } R))$

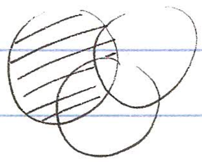
$\equiv P \text{ AND } ((\text{NOT } Q) \text{ OR } (\text{NOT } R))$

$\equiv (P \text{ AND } (\text{NOT } Q)) \text{ OR } (P \text{ AND } (\text{NOT } R))$ □.

c)

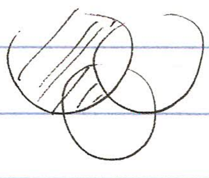


$B \cap C$

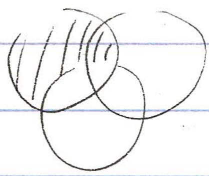


$A \setminus (B \cap C)$

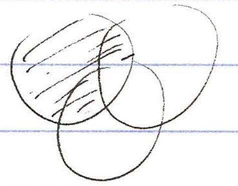
=



$A \setminus B$



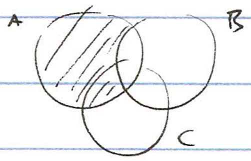
$A \setminus C$



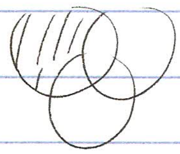
$(A \setminus B) \cup (A \setminus C)$.

✓.

16. a.

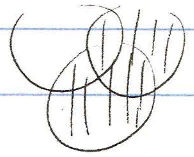


$A \setminus B$

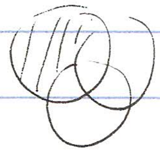


$(A \setminus B) \setminus C$

|| ✓

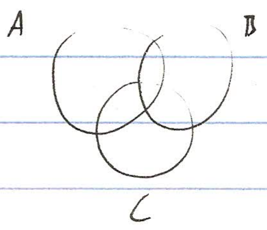


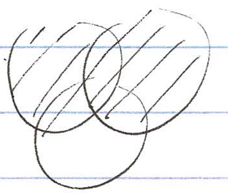
$B \cup C$



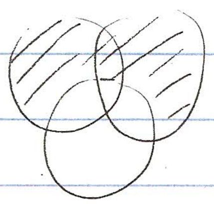
$A \setminus (B \cup C)$

b.

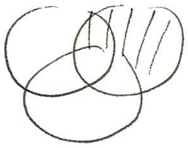




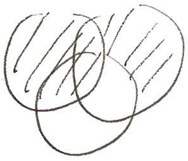
$A \cup B$



$A \cup B \setminus C$



$B \setminus C$



$A \cup (B \setminus C)$

See $(A \cup B) \setminus C \neq A \cup (B \setminus C)$ unless $A \cap C = \emptyset$.

17. $\binom{4+7}{4} = \binom{11}{4} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4!} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{24} = 11 \cdot 10 \cdot 3 = 330.$

18. $\frac{\binom{8}{4}}{2^8} = \frac{\frac{8 \cdot 7 \cdot 6 \cdot 5}{4!}}{256} = \frac{70}{256} = \frac{35}{128} = 0.273..$

19.

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - \underbrace{(|A \cap B| + |A \cap C| + .. + |C \cap D|)}_{\text{all pairwise intersections}}$$

$$= 4 \cdot 51 - \binom{4}{2} \cdot 1$$

note triple intersections are empty here!

$$= 204 - 6 = 198.$$

$\therefore$ Probability $= \frac{198}{\binom{52}{2}} = \frac{198}{\frac{52 \cdot 51}{2}} = \frac{198}{26 \cdot 51} = \frac{198}{1326} = 0.149..$

20. $|A_1 \cup .. \cup A_6| \overset{IEP}{=} \sum_{i=1}^{6} |A_i| - \sum_{1 \le i < j \le 6} |A_i \cap A_j| + .. + (-1)^5 |A_1 \cap .. \cap A_6|$

$|A_i| = 5^8, \quad |A_i \cap A_j| = 4^8$, etc.

so $|A_1 \cup .. \cup A_6| = 6 \cdot 5^8 - \binom{6}{2} \cdot 4^8 + \binom{6}{3} \cdot 3^8 - \binom{6}{4} \cdot 2^8$

$$+ \binom{6}{5} \cdot 1^8 - \binom{6}{6} \cdot 0^8$$

$$= 6 \cdot 5^8 - 15 \cdot 4^8 + 20 \cdot 3^8 - 15 \cdot 2^8 + 6$$

~~= 45936~~ $= 1488096$

$\therefore$ Probability $=$ ~~$\frac{6^6 - 45936}{6^6}$~~ $\frac{6^8 - 1488096}{6^8} = 0.114..$