Math 235 Syllabus Fall 2013

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Numbers refer to the course text Bretscher, Linear algebra with applications, 5th ed.

(1.1) *Introduction to linear systems*: Solving systems of linear equations, geometric interpretation, number of solutions.

(1.2) *Matrices, Vectors, and Gauss-Jordan elimination*: Matrix and vector notation, Gaussian elimination. Reduced row echelon form of a matrix.

(1.3) *Solutions of linear systems; Matrix algebra*: Number of solutions of a linear system. Dot product $\mathbf{x} \cdot \mathbf{y}$ of vectors. Linear combinations of vectors, product $A\mathbf{x}$ of a matrix A and a vector $\mathbf{x}$. Matrix form $A\mathbf{x} = \mathbf{b}$ of linear system. Rank of a matrix in terms of its row echelon form.

(2.1) *Linear transformations and their inverses*: The linear transformation

$$T\colon \mathbb{R}^n \to \mathbb{R}^m, \quad T(\mathbf{x}) = A\mathbf{x},$$

associated to a $m \times n$ matrix A. The columns of A are the vectors $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ (the images of the standard basis vectors under T). A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear iff $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

(2.2) *Linear transformations in geometry*: Scaling, reflection, rotation, orthogonal projection, shears.

(2.3) *Matrix product*: Definition of matrix product AB. The matrix product corresponds to composition of linear transformations. Matrix multiplication is not commutative ($AB \neq BA$ in general), but is associative ($A(BC) = (AB)C$) and satisfies the distributive laws.

(2.4) *Inverse of a linear transformation*: Inverse of a function in general. If a matrix is invertible then it is square. An $n \times n$ matrix is invertible if and only if $\operatorname{rank}(A) = n$. Computation of the inverse of an $n \times n$ matrix A using row operations applied to the $n \times 2n$ matrix $(A\, I)$. Inverse of a product: $(AB)^{-1} = B^{-1}A^{-1}$. Inverse of a 2×2 matrix: the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible iff the determinant $\det A := ad - bc$ is nonzero, and then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(3.1) *Image and kernel of a linear transformation*: Image of a function in general. Span of a set of vectors. The image of the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = A\mathbf{x}$ is the span of the columns of the matrix A. Definition of the kernel of a linear transformation. $T(\mathbf{x}) = T(\mathbf{y})$ iff $\mathbf{x} - \mathbf{y}$ lies in the kernel.

(3.2) *Subspaces of $\mathbb{R}^n$; Bases and linear independence*: Definition of a subspace of $\mathbb{R}^n$. Examples and non-examples. For a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ the kernel is a subspace of the domain $\mathbb{R}^n$ and the image is a subspace of the codomain $\mathbb{R}^m$. Linearly independent sets. Basis of a subspace. Unique representation of elements of a subspace in terms of a basis. Computation of a basis of the kernel and image of a linear transformation from the row echelon form of the corresponding matrix.

(3.3) *The dimension of a subspace of* $\mathbb{R}^n$: Two bases of a subspace have the same number of elements. Dimension of a subspace. The rank of a matrix A is the dimension of the image of the associated linear transformation. The rank–nullity theorem.

(3.4) *Coordinates*: Coordinates of a vector $\mathbf{x} \in \mathbb{R}^n$ with respect to a basis $\mathcal{B}$ of $\mathbb{R}^n$. Matrix of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ with respect to a basis $\mathcal{B}$ of $\mathbb{R}^n$. Similar matrices.

(4.1) *Linear spaces (Vector spaces)*: Definition of a vector space. Examples. Generalization of notions from subspaces of $\mathbb{R}^n$ to general vector spaces. Infinite dimensional vector spaces.

(4.2) *Linear transformations and isomorphisms*: Generalizations of notions from linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$ to linear transformations of vector spaces $T\colon V \to W$. Criteria for $T\colon V \to W$ to be an isomorphism in terms of $\dim V$, $\dim W$, $\operatorname{rank}(T)$, and $\ker(T)$.

(4.3) *The matrix of a linear transformation*: Matrix of a linear transformation $T\colon V \to V$ from a vector space V to itself with respect to a basis $\mathcal{B}$ of V. Change of basis.

(6.1) *Introduction to determinants*: Elementary computation of 3×3 determinant. Definition of $n \times n$ determinants via patterns (permutations). Determinant of a triangular matrix.

(6.2) *Properties of the determinant*: The transpose A^T of a matrix A. $\det(A^T) = \det(A)$. Linearity of the determinant in the rows and columns. Computation of determinant via row and column operations. $\det(AB) = \det(A)\det(B)$. A square matrix A is invertible iff $\det(A) \neq 0$.

(6.3) *Geometrical interpretation of the determinant; Cramer's rule*: Geometric interpretation of the determinant of an $n \times n$ matrix for $n = 2$ and $n = 3$.

(7.1) *Diagonalization*: Eigenvectors, eigenvalues, and diagonalization. Examples.

(7.2) *Finding the eigenvalues of a matrix*: The characteristic equation

$$\det(A - \lambda I) = 0.$$

The 2×2 case. (The trace of a matrix.) Eigenvalues of a triangular matrix. The algebraic multiplicity of an eigenvalue.

(7.3) *Finding the eigenvectors of a matrix*: The eigenspace associated to an eigenvalue. The geometric multiplicity of an eigenvalue. The geometric multiplicity is less than or equal to the algebraic multiplicity. An $n \times n$ matrix A is diagonalizable iff there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. Algorithm for diagonalization of a matrix.

(7.4) *More on dynamical systems*: Computing powers of a matrix via diagonalization.

(7.5) *Complex eigenvalues*: Complex numbers. Polar form and geometric interpretation of multiplication. Complex eigenvalues and eigenvectors. A 2×2 matrix with complex eigenvalues is similar to a rotation-scaling matrix. Fundamental theorem of algebra (statement only). The number of complex eigenvalues (counted with algebraic multiplicities) of an $n \times n$ matrix A equals n.

(5.1) *Orthogonal projections and orthonormal bases*: Orthogonal vectors in $\mathbb{R}^n$. Orthonormal bases. Orthogonal projection onto a subspace.

(5.2) *Gram–Schmidt process and QR factorization*: Gram–Schmidt construction of an orthonormal basis of a subspace of $\mathbb{R}^n$.