MANIFOLDS 703 EXAM (ANSWERS)

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems.

You may cite results proved in class, in homework (from Problem sets I and II), or in the text in your solutions to the problems below, and need not reprove them. Otherwise (with the exception of Problem 1) you must provide complete proofs.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

These are only sketches of solutions; complete solutions require more details. You should be able to work out all these problems, even the ones you didn't try on the exam.

(1) Please provide the following definitions and statements. **ANS:** Omitted, since they are in the textbook.
   (a) State the inverse function theorem.
   (b) Define *immersion* and *submersion*.
   (c) Define *immersed submanifold* and *embedded submanifold*.
   (d) Define *Lie group*.

(2) Let X, Y be smooth manifolds and let $f\colon X \to Y$ be a smooth map. Define $Z = \{(x, f(x))\} \subset X \times Y$. Show that the map $F\colon X \to Z$ given by $F(x) = (x, f(x))$ is a diffeomorphism. **ANS:** It's not hard to see that F is smooth by checking in local coordinates. The inverse map is given by $F^{-1}(x, f(x)) = x$, which is also smooth. Hence F is a diffeomorphism.

(3) Let $X \subset \mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$ be defined by the equation $\sum_{i=0}^n x_i y_i = 0$, where the x_i are homogeneous coordinates on the first factor, and the y_i are homogeneous coordinates on the second factor. Show that X is an embedded submanifold. What is the dimension of X? **ANS:** Use the fact that if $X \cap U$ is an embedded submanifold for a covering $\{U\}$ of $Y = \mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$, then X is embedded. By symmetry it suffices to look in the two standard charts $U_1 = ((1, x_1, \dots, x_n), (1, y_1, \dots, y_n))$ and $U_2 = ((1, x_1, \dots, x_n), (y_0, 1, y_2, \dots, y_n))$. For U_1, we have the map $f\colon U_1 \to \mathbb{R}$ given by $((1, x_1, \dots, x_n), (1, y_1, \dots, y_n)) \mapsto 1 + x_1 y_1 + \cdots + x_n y_n$, and $X \cap U = f^{-1}(0)$. This is a submersion on $X \cap U$, which means $X \cap U$ is an regular submanifold of codimension 1. The chart U_2 is similar. Hence X is embedded, and has codimension 1 = dimension $n - 1$.

(4) Prove or give a counterexample: Suppose X is a manifold with a collection of charts $\{(U, \varphi_U)\}$. Suppose $f\colon X \to Y$ is a smooth map to a manifold Y such that

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$f(X \cap U)$ is an embedding for each U. Then f is an embedding of X into Y. **ANS:** This is false; it sounds similar to a homework problem, but it's not the same statement. A counterexample is given by the immersed figure 8. We can cover it by three open sets, say the interval $(-1, 1)$ about 0, and two open sets $(1-\varepsilon, \infty)$, $(-\infty, -1+\varepsilon)$. It's easy to see that these are embedded in $\mathbb{R}^2$ by the map that takes $\mathbb{R}$ to the figure 8, but the figure 8 isn't a regular submanifold.

(5) Let $f\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ be defined by $[s : t] \mapsto [s^n : s^{n-1}t : \cdots : st^{n-1} : t^n]$. Show that this is an embedding of $\mathbb{P}^1(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{R})$. **ANS:** We can use the fact that a one-one immersion $f\colon M \to N$ from a compact manifold M is an embedding, since $M = \mathbb{P}^1(\mathbb{R})$ is compact. There are two charts U_0, U_1 covering M (in fact $M \simeq S^1$, and our projective charts are the same as the stereographic charts). Verifying 1-1 immersion on these charts is easy, so f is an immersion. All points of M appear in U_1 except $[1 : 0]$, but $[1 : 0]$ is taken to $[1 : 0 : \cdots : 0]$ and no point of U_1 is. Thus f is injective.

(6) Let $M_2(\mathbb{R})$ be the set of 2×2 real matrices, and let $X \subset M_2(\mathbb{R})$ be the matrices of rank exactly one (*not* ≤ 1). Show that X is a smooth submanifold of $M_2(\mathbb{R})$. What is its dimension? **ANS:** Let $f\colon M_2(\mathbb{R}) \to \mathbb{R}$ be the determinant map. Show it's a submersion on X. (Note that if we added the zero matrix by considering rank ≤ 1, this wouldn't be a submersion.) Hence X is a smooth submanifold of codimension 1 = dimension 3.

(7) Let $X \subset \mathbb{R}^3$ be the set of points (x, y, z) satisfying $(\sqrt{x^2+y^2} - 2)^2 + z^2 = 1$.
   (a) Show that X is a regular submanifold of $\mathbb{R}^3$.
   (b) What is the dimension of X?
   (c) X is a manifold you've seen before. What is it? **ANS:** Taking cylindrical coordinates r, θ, z in $\mathbb{R}^3$, we see that X is the solution of $(r-2)^2 + z^2 = 1$. So to make X we take a circle with radius 1 centered at $(2, 0, 0)$, which we then rotate about the z-axis. Hence X is a torus. This means one way to prove X is a manifold is by using charts. We can take two, corresponding to the two stereographic charts of the S^1 we rotate. Alternatively, we could use the submersion theorem by showing that $Df|_X$ has rank 1, where $f(x, y, z) = (\sqrt{x^2+y^2} - 2)^2 + z^2$.

(8) (a) Prove or give a counterexample: Let H be a closed Lie subgroup of a Lie group G. Let H act on G by left translations: $(h, g) \mapsto hg$, where $h \in H$, $g \in G$. Then this action is always free. **ANS:** This is true. If $hg = g$, then $h = e$ since you can cancel g.
   (b) Show that if a Lie group G acts transitively on a set S, and $x, y \in S$, then the stabilizer subgroup of x is isomorphic to the stabilizer subgroup of y. **ANS:** Let $x, y \in S$, and suppose $gx = y$. Then if H is the stabilizer subgroup of x, the subgroup gHg^{-1} is the stabilizer of y. These groups are clearly isomorphic.

(9) (a) Let G be the set of real 3×3 upper-triangular matrices with 1s on the diagonal. Show that G is a Lie group under matrix multiplication. **ANS:** This is a subgroup of $GL_3(\mathbb{R})$, and by explicit computation we can see that multiplcation and inversion are polynomial functions in the entries of any $g \in G$.

(b) Let $H \subset G$ be the subgroup of matrices with integral entries. Show that G/H is a smooth manifold. **ANS:** We have to show that H is a discrete subgroup of G. This is easy to see using the diffeomorphism $G \simeq \mathbb{R}^3$, which takes H to $\mathbb{Z}^3$. (Note that $\mathbb{R}^3$ is a Lie group under addition, and this diffeomorphism isn't a group isomorphism. In particular the quotient G/H is not a 3-torus.) Then a theorem quoted in class implies that G/H is a smooth manifold. Alternatively, we can argue directly by showing that the action of H on G is free and properly discontinuous.