MATH 703 — FALL 2004
ANSWERS TO PROBLEM SET I

These are only sketches of solutions. Full solutions require more details.

Problem 1. Show that $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ if $x \leq 0$ and $\exp(-1/x^2)$ if $x > 0$ is smooth but not real analytic. **ANS**: Show that if $x > 0$, then $f^{(k)}(x) = P(x^{-1})\exp(-1/x^2)$, where $P(z)$ is polynomial in z. Then prove $\lim_{x\to 0^+} f^k(x) = 0$.

Problem 2. Let $X = \mathbb{R} \times \{0, 1\}$ with the usual product topology. Define an equivalence relation on X by $(x, 0) \sim (x, 1)$ if and only if $x > 0$. Let $Y = X/\!\sim$ with the quotient topology. Prove Y is locally euclidean but is not Hausdorff. **ANS**: The two funny points are the images of $(0, 0)$ and $(0, 1)$. These points are distinct in Y, yet any open set around the first must meet any open set around the second somewhere to the left. However, Y is locally Euclidean since these open sets are homeomorphic to open intervals in $\mathbb{R}$.

Problem 3. Give a complete proof that the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is a smooth manifold using stereographic projection.**ANS**: The maps from the two chart to $\mathbb{R}^{n-1}$ can be found using elementary geometry. The main point is to show that the transition function between the images of the intersection is inversion in the unit sphere in $\mathbb{R}^{n-1}$: $x \mapsto x/|x|^2$.

Problem 4.

(1) Let $\mathbb{P}^n(\mathbb{C})$ be the quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$, where the multiplicative group $\mathbb{C}^\times$ acts diagonally. Prove that $\mathbb{P}^n(\mathbb{C})$ is a smooth manifold. What is its dimension? **ANS**: The proof follows exactly the same stryucture as in the case of $\mathbb{P}^n(\mathbb{R})$. The only difference is that the functions are now from $\mathbb{C}^n$ to $\mathbb{C}^n$. Now $\mathbb{C}^n$ can be regarded as $\mathbb{R}^{2n}$ via $(\cdots, z_i, \dots) \mapsto (\cdots, \Re z_i, \Im z_i, \dots)$, so you must verify that when the transition functions are written out in terms of the coordinates on $\mathbb{R}^{2n}$, they are diffeomorphisms.

(2) Let $\mathbb{H}$ be the quaternions, that is

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\},$$

where addition is defined componentwise, and multiplication is determined by the rules

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k.$$

Recall that $\mathbb{H}$ is a division algebra (all nonzero elements are invertible), and multiplication is associative but not commutative. Define $\mathbb{P}^n(\mathbb{H})$, prove it's a smooth manifold, and compute its dimension. (Hint: generalize the proofs for $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$, but beware that $\mathbb{H}$ isn't commutative.) **ANS**: Here the tricky part arises in the proof that $\mathbb{P}^n(\mathbb{H})$ is hausdorff. I had thought a simple modification of the proof for $\mathbb{P}^n(\mathbb{R})$ would work, but I was mistaken. (It ends up being rather complicated to show using our current methods that the quotient is hausdorff.) But if one assumes that the quotient is hausdorff, then again the main work is to verify that the transition functions, in terms of the coordinates on $\mathbb{R}^{4n}$, are diffeomorphisms.

Problem 5. Give a complete proof that the grassmannian $G(2,4)$ is a smooth manifold. **ANS**: There are 6 charts U_I, where I ranges over all 2-element subsets of $\{1,2,3,4\}$. By symmetry there are two types of transition functions to consider: those arising from $U_I \cap U_J$ where $I \cap J$ is a singleton, and those where $I \cap J = \varnothing$. The first was done in class. The second involves matrix inversion, which is a diffeomorphism.

Problem 6. Regard the sphere S^{2n-1} as a subset of $\mathbb{C}^n$, by taking it to be the set of complex vectors $(z_1,\dots,z_n)$ satisfying $|z_1|^2+\cdots+|z_n|^2=1$. Define an action of $\mathbb{R}$ on $\mathbb{C}^n$ by

$$(t, z_1, \dots, z_n) \longmapsto (\exp(it)z_1, \dots, \exp(it)z_n).$$

(1) Verify that this action preserves S^{2n-1}. Given a point $z \in S^{2n-1}$, what is its orbit $O(z)$? **ANS**: After multiplying all the z_i by a complex number of norm one, the defining equation of S^{2n-1} is still satisfied. The orbits are circles, since they are compact 1-dimensional submanifolds. In fact one can explicitly write a diffeomorphism from any orbit to S^1.

(2) Show that the relation $z \sim z'$ if $O(z) = O(z')$ is an open equivalence relation, and that the quotient $S^{2n-1}/\sim$ is a smooth manifold. **ANS**: Very similar to other proofs we have given in class.

(3) The quotient is actually a manifold you know. What is it? **ANS**: It's $\mathbb{P}^{n-1}(\mathbb{C})$. To see this, note that $\mathbb{C}^\times \simeq \mathbb{R}^\times \times S^1$. So we are really modding out by $\mathbb{C}^n$ first by $\mathbb{R}^\times$ to get the unit sphere, then by S^1 to get the final quotient.

(4) (Extra credit) In the case of $n = 2$, the big sphere is S^3. We can draw pictures of $S^3 \smallsetminus \{(0,i)\}$ using stereographic projection to map it to $\mathbb{R}^3$ (the map will take $(z_1,z_2) \in S^3$ to $(x,y,z) \in \mathbb{R}^3$ by formulas you wrote down in Problem 3). Draw the orbits of the S^1 action as subsets of $\mathbb{R}^3$. (Hint: one will be the unit circle in the xy plane, and one will be the z axis.) **ANS**: No one tried this! That's too bad. Anyway, the circles fill out concentric tori in S^3. Try googling for *hopf fibration*. R. Scharein's site has a nice animation of how the circles fit together. These cicles on the torus are called *Villarceau circles.*

Problem 7. .

(1) Show that the product of two smooth manifolds is a smooth manifold, with dimension given by the sum of the dimensions of the factors. **ANS**: The charts on the product have the form $(U \times V, \varphi \times \theta)$, where (U,φ), (V,θ) are part of atlases of the factors.

(2) Let $T^n \subset \mathbb{C}^n$ be the image of the map $f \colon \mathbb{R}^n \to \mathbb{C}^n$ that takes $(x_1, \ldots, x_n)$ to $(\exp(ix_1), \ldots, \exp(ix_n))$. Show that T_n is a smooth manifold of dimension n. **ANS**: T^n is the same as $(S^1)^n$, as this description shows.

Problem 8. Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3 + xy + y^3 + 1$.

(1) Find the differential[1] Df_p, where $p = (a, b) \in \mathbb{R}^2$.
(2) For which points $p = (a, b) \in \mathbb{R}^2$ is Df_p injective? onto? **ANS**: Omitted.

Problem 9. Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x, y) \mapsto (x^2+y^2, x^2-y^2)$. Find the subsets of the domain of f of constant rank. **ANS**: If x, y are both nonzero the rank is 2. If both are zero the rank is 0. Otherwise the rank is 1.

Problem 10. Let $F \colon \mathbb{R}^3 \to \mathbb{R}^5$ be the map:

$$F(x, y, z) \;=\; (x^2, xy, xz + y^2, yz, z^2).$$

Find all points $p \in \mathbb{R}^3$ where the differential DF_p has rank 3, 2, 1, or 0. **ANS**: DF has rank 0 at the origin, and rank 3 otherwise.

Problem 11.

(1) Show that $f\colon \operatorname{GL}_n(\mathbb{R}) \to \operatorname{GL}_n(\mathbb{R})$ given by $m \mapsto m^2$ is smooth. (Hint: you may use the following characterization of C^1 functions. A function $f \colon \mathbb{R}^k \to \mathbb{R}^\ell$ is C^1 at $a \in \mathbb{R}^k$ if each partial derivative $\partial f_j/\partial x_i$ is defined in an open neighborhood of a and is C^0 at a). **ANS**: The entries of m^2 are polynomial functions of the entries of m. Thus the various partial derivatives are certainly C^0, so the map is C^1. Now use induction to show C^∞.
(2) Show that there exist open neighborhoods U, V of the identity matrix in $\operatorname{GL}_n(\mathbb{R})$ such that $f \colon U \to V$ is invertible, i.e. if $m \in V$ then a matrix square root can be computed. (*Remark:* computationally the square root of m can be computed using the binomial theorem, but this isn't useful for the proof!) **ANS**: Use the inverse function theorem. The main point is to show by explicit computation that Df has rank n^2 at the identity.

[1] *aka* the Jacobian