Solutions of Problem #4, Midterm Exam 1, Calculus, Math 233

Suppose that $\mathbf{r}(t)$ has derivative $\mathbf{r}'(t) = \langle -\sin 2t, \cos 2t, 0\rangle$ on the interval $0 \le t \le 1$. Suppose we know that $\mathbf{r}(0) = \langle \frac{1}{2}, 0, 1\rangle$.
(a) Determine $\mathbf{r}(t)$ for all t.
(b) Show that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all t.
(c) Find the arclength of the graph of the vector function $\mathbf{r}(t)$ on the interval $0 \le t \le 1$.

Solution of part **(a)**:
The position function $\mathbf{r}(t)$ can be found as

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle -\sin 2t, \cos 2t, 0\rangle\,dt$$
$$= \left\langle \frac{1}{2}\cos 2t + c_1, \frac{1}{2}\sin 2t + c_2, c_3 \right\rangle.$$

To find the constants compute $\mathbf{r}(0)$:

$$\mathbf{r}(0) = \left\langle \frac{1}{2} + c_1, c_2, c_3 \right\rangle = \left\langle \frac{1}{2}, 0, 1 \right\rangle,$$

therefore $c_1 = 0$, $c_2 = 0$, $c_3 = 1$, and

$$\mathbf{r}(t) = \left\langle \frac{1}{2}\cos 2t, \frac{1}{2}\sin 2t, 1 \right\rangle.$$

Another method is to use the formula

$$\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{r}'(u)\,du,$$

which gives the same answer.

Grading of part **(a)***:*
-2 points if a student forgot to subtract $1/2$ from $1/2\cos 2t$ in the first component of $r(t)$, or there is a mess in constants (function is multiplied by 2 instead of $1/2$, etc.).

Solution of part **(b)***:*
Two vectors are orthogonal if their scalar product is equal to zero. Hence,

$$\mathbf{r}'(t)\cdot\mathbf{r}(t) = \langle -\sin 2t, \cos 2t, 0\rangle \cdot \left\langle \frac{1}{2}\cos 2t, \frac{1}{2}\sin 2t, 1 \right\rangle$$
$$= -\frac{1}{2}\sin 2t\cdot\cos 2t + \frac{1}{2}\cos 2t\cdot\sin 2t + 0\cdot 1 = 0.$$

Grading of part **(b)***:*
-5 points for the attempt to solve using the vector product
-4 points for the attempt to find value of the scalar product at specific point, not for all t
-3 points if scalar product is confused with vector components

Solution of part **(c)***:*
The arc length of the part of the curve is

$$L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 \sqrt{(-\sin 2t)^2 + (\cos 2t)^2 + 0^2}\, dt = \int_0^1 1\, dt = 1.$$

Grading of part **(c)***:*
-4 points if $\cos^2 2t + \sin^2 2t = 2$, or similar mistake ($\cos^2 2t = \cos 4t$, etc.)
-3 points if the integral for the arc length is found together with correct integrand, but the remaining part is not satisfactory
-2 for the mess in limits, attempts to solve numerically

Each part of the problem #4 cost **-6** to **-8** points if everything was solved not correct. Also, **-1** for the general mess/or no comprehensive explanation/minor mistake in constants.