3. (a) A *plane* curve is given by the graph of the vector function

$$\mathbf{u}(t) = \langle 1 + \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi.$$

Find a single equation for the curve in terms of x and y, by eliminating t.
(b) Consider the *space* curve given by the graph of the vector function

$$\mathbf{r}(t) = \langle 1 + \cos t, \sin t, t \rangle, \quad 0 \le t \le 2\pi.$$

Sketch the curve and indicate the direction of increasing t in your graph.
(c) Determine parametric equations for the line T tangent to the graph of the *space* curve for $\mathbf{r}(t)$ at $t = \pi/3$, and sketch T in the graph obtained in part **(b)**.

Graded by Roman Fedorov

Solution. (a) We have $x(t) = 1 + \cos t$, $y(t) = \sin t$. Thus $x - 1 = \cos t$. Since $\cos^2 t + \sin^2 t = 1$, we obtain $(x-1)^2 + y^2 = 1$.

(b) The equation $(x-1)^2 + y^2 = 1$ is the equation of the cylinder of radius 1, whose center is the vertical line through $(1, 0, 0)$. Thus the curve is one turn of the helix on this cylinder. It goes upward, starting at $(2, 0, 0)$, ending at $(2, 0, 2\pi)$.

(c) We obtain: $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$. Thus the vector

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1 \right\rangle$$

gives the direction of the line. We also need a point on the line:

$$\mathbf{r}\left(\frac{\pi}{3}\right) = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right).$$

The vector equation of the line is given by

$$\mathbf{r}\left(\frac{\pi}{3}\right) + t\mathbf{r}'\left(\frac{\pi}{3}\right) = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2} + \frac{1}{2}t, \frac{\pi}{3} + t\right).$$

Hence the parametric equation of the line is:

$$x = \frac{3}{2} - \frac{\sqrt{3}}{2}t, \qquad y = \frac{\sqrt{3}}{2} + \frac{t}{2}, \qquad z = \frac{\pi}{3} + t.$$

Grading Scheme. (a) was graded out of **6** points. The full credit was given for correct answer. The common answers $x = 1 + \cos(\sin^{-1} y)$ and $y = \sin(\cos^{-1} x - 1)$ received just **2** points, because this gives only a part of a curve. For example, for the first answer, we have $-\pi/2 \le \sin^{-1} y \le \pi/2$, thus $x \ge 1$, and we get only one half of the circle.

(b) was graded out of **6** points. I was giving points as long as the picture looked helix to me. If the graph was a circle **0** points were given. I was taking off **1** point each for wrong direction, and a wrong center of the helix.

The most common mistake was to have more than one turn (many turns, two turns, or $1\frac{1}{2}$ turns), **1** point was taken off in those cases. Thus, for a helix-like picture students could receive from **3** to **6** points.

(c) It was graded out of **8** points as follows:
2 points if $\mathbf{r}'(t)$ was calculated correctly.
2 points for calculating $\mathbf{r}'(\pi/3)$ correctly.
1 point for calculating $\mathbf{r}(\pi/3)$.

2 points for writing out the parametric equation.
1 point for sketching the tangent.

I was taking **1** point off if $\sin(\pi/3)$ or $\cos(\pi/3)$ was left in the answer.

The most common mistake was to forget to evaluate $\mathbf{r}'(\pi/3)$, or to use for the vector equation of the line $\mathbf{r}'(\pi/3) + \mathbf{r}'(t)$. In each of those cases the resulting equation was **not** an equation of the line.