DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

MATH 233

EXAM 1

Spring 2007

NAME:______________________________

Section Number:____________________
Instructor's Name: ____________________

In problems that require reasoning or algebraic calculation, it is not sufficient just to write the answers. You must explain how you arrived at your answers, and show your algebraic calculations.

You can **leave answers in terms of fractions and square roots**, but if approximate numerical answers are used, they should be rounded off to 4 significant figures.

$\langle x, y, z \rangle$, $[x, y, z]$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; are all permissible notations for the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

1.	(20)	________
2.	(20)	________
3.	(20)	________
4.	(20)	________
5.	(20)	________
Total		________

Perfect Paper $\longrightarrow$ 100 Points.

There are seven pages, including this one, in this exam and five problems. Make sure you have them all before you begin!

FORMULAS

$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$

$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta =$ area of parallelogram with sides $\mathbf{a}$, $\mathbf{b}$.

The vector projection of $\mathbf{b}$ onto (in the direction of)

$\mathbf{a}$ is $proj_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$

The scalar projection (component) of $\mathbf{b}$ onto

$\mathbf{a}$ is $comp_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

The volume of the “box” determined by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$

Arc length of parametrized curve:

$$L = \int_a^b |\mathbf{r}'(t)|\,dt.$$

1. a) (10 points) Find parametric equations for the line which contains $A(7, 6, 4)$ and $B(4, 6, 5)$.

A vector parallel to the line is $\langle 7-4, 6-6, 4-5\rangle = \langle 3, 0, -1\rangle$.
A point on the line is $A(7, 6, 4)$.
Therefore parametric equations for the line are:

$$\begin{aligned} x &= 7 + 3t \\ y &= 6 \\ z &= 4 - t \end{aligned}$$

b) (10 points) Find the parametric equations for the line of intersection of the planes $x - 2y + z = 5$ and $2x + y - z = 0$.

A vector parallel to the line is the cross product of the normal vectors of the planes:

$$\begin{aligned} \mathbf{v} &= \langle 1, -2, 1\rangle \times \langle 2, 1, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \mathbf{k} \\ &= 1\mathbf{i} - (-3)\mathbf{j} + 5\mathbf{k} \\ &= \langle 1, 3, 5\rangle \end{aligned}$$

A point on the line is *any* (x_0, y_0, z_0) that satisfies *both* of the plane equations:$(1, -2, 0)$
Therefore parametric equations for the line are:

$$\begin{aligned} x &= 1 + t \\ y &= -2 + 3t \\ z &= 5t \end{aligned}$$

2. a) (8 points) Find an equation of the plane which contains the points $P(-1,0,2)$, $Q(1,-2,1)$ and $R(2,0,-1)$.

A normal vector to the plane can be found by taking the cross product of *any* two vectors that lie *in* the plane. Two vectors that lie in the plane are $PQ\langle 2,-2,-1\rangle$ and $PR\langle 3,0,-3\rangle$. So the normal vector is

$$\begin{aligned}
\mathbf{n} &= \langle 2,-2,-1\rangle \times \langle 3,0,-3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 3 & 0 & -3 \end{vmatrix} \\
&= \begin{vmatrix} -2 & -1 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} \mathbf{k} \\
&= 6\mathbf{i} - (-3)\mathbf{j} + 6\mathbf{k} \\
&= \langle 6,3,6\rangle
\end{aligned}$$

A point on the plane is $P(-1,0,2)$.
Therefore an equation of the plane is

$$6(x-(-1)) + 3(y-0) + 6(z-2) = 0 \tag{1}$$

or, simplified,

$$2x + y + 2z - 2 = 0 \tag{2}$$

b) (6 points) Find the distance from the point $(1,0,-1)$ to the plane $2x + y - 2z = 1$.

Using the distance formula for the distance between a point (x_0, y_0, z_0) and a plane $ax + by + cz + d = 0$,

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{a^2 + b^2 + c^2} = \frac{|(2)(1) + (1)(0) + (-2)(-1) - 1|}{\sqrt{2^2 + 1^2 + (-2)^2}} = 1$$

c) (6 points) Find the point P in the plane $2x+y-2z = 1$ which is closest to the point $(1,0,-1)$. (Hint: You can use part b) of this problem to help find P or first find the equation of the line passing through P and the point $(1,0,-1)$ and then solve for P.)

First, get the equation of the line that goes through the point $(1,0,-1)$ that is normal to the plane:

$$\begin{aligned}
x &= 1 + 2t \\
y &= t \\
z &= -1 - 2t
\end{aligned}$$

Now, the point in the plane which is closest to $(1,0,-1)$ is the intersection of this line and the plane. To find this intersection, substitute the line equations into the plane equation:

$$2(1+2t)+(t)-2(-1-2t)=1$$

Simplifying and solving for t,

$$9t+4=1 \Rightarrow t=-\frac{1}{3}$$

Plugging this t-value into the line equation, we get the coordinates of the point of intersection:

$$\begin{aligned} x &= 1+2(-\frac{1}{3})=\frac{1}{3} \\ y &= -\frac{1}{3} \\ z &= -1-2(-\frac{1}{3})=-\frac{1}{3} \end{aligned}$$

So, the point on the plane closest to $(1,0,-1)$ is $(\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$.

3. a) (10 points) Consider the two space curves

$$\mathbf{r}_1(t) = \langle \cos(t-1), t^2 - 1, 2t^4 \rangle, \quad \mathbf{r}_2(s) = \langle 1 + \ln s, s^2 - 2s + 1, 2s^2 \rangle,$$

where t and s are two independent real parameters. Find the cosine of the angle between the tangent vectors of the two curves at the intersection point $(1, 0, 2)$.

The point $(1, 0, 2)$ corresponds to the t-value $t = 1$ for $\mathbf{r}_1$ and s-value $s = 1$ for $\mathbf{r}_2$.
The tangent vector to $\mathbf{r}_1$, at $t = 1$, is

$$\mathbf{r}_1'(t) = \langle \sin(t-1), 2t, 8t^3 \rangle = \langle \sin(1-1), 2(1), 8(1^3) \rangle = \langle 0, 2, 8 \rangle$$

Therefore the tangent line to $\mathbf{r}_1$, which goes through the point $(1, 0, 2)$ and is parallel to the tangent vector $\langle 0, 2, 8 \rangle$ is

$$\begin{aligned} x &= 1 \\ y &= 2t \\ z &= 2 + 8t \end{aligned}$$

The tangent vector to $\mathbf{r}_2$, at $s = 1$, is

$$\mathbf{r}_2'(t) = \langle \frac{1}{s}, 2s - 2, 4s \rangle = \langle \frac{1}{1}, 2(1) - 2, 4(1) \rangle = \langle 1, 0, 4 \rangle$$

Therefore the tangent line to $\mathbf{r}_2$, which goes through the point $(1, 0, 2)$ and is parallel to the tangent vector $\langle 1, 0, 4 \rangle$ is

$$\begin{aligned} x &= 1 + t \\ y &= 0 \\ z &= 2 + 4t \end{aligned}$$

b) (10 points) Find the center and radius of the sphere $x^2 + y^2 + 2y + z^2 + 4z = 20$.

Completing the square in the y and z variables,

$$x^2 + (y^2 + 2y + 1) + (z^2 + 4z + 4) = 20 + 1 + 4$$

Rewriting,

$$x^2 + (y + 1)^2 + (z + 2)^2 = 25 = 5^2$$

Therefore the center is $(0, -1, -2)$ and the radius is 5.

4. The velocity vector of a particle moving in space equals

$$\mathbf{v}(t) = 2t\mathbf{i} - 2t\mathbf{j} + t\mathbf{k} \text{ at any time } t \geq 0 \tag{3}$$

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a)(6 points) At the time $t = 4$, this particle is at the point $(0, 5, 4)$. Find an equation of the tangent line to the curve at the time $t = 4$.

This line goes through the point $(0, 5, 4)$ and has parallel tangent vector $\mathbf{v}(4) = \langle 8, -8, 4 \rangle$, so has parametric equations

$$\begin{aligned} x &= 8t \\ y &= 5 - 8t \\ z &= 4 + 4t \end{aligned}$$

b) (6 points) Find the length of the arc traveled from time $t = 2$ to time $t = 4$.

Using the arclength formula,

$$\begin{aligned} \int_2^4 |v(t)|\,dt &= \int_2^4 \sqrt{(2t)^2 + (-2t)^2 + t^2 dt} \\ &= \int_2^4 \sqrt{9t^2} dt \\ &= \int_2^4 3t dt \\ &= \frac{3}{2} t^2 \Big|_2^4 \\ &= \frac{3}{2}(16 - 4) \\ &= 18 \end{aligned}$$

c) (8 points) Find a vector function which represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $3x - 2y + z = 2$.

Since the first equation is the equation of a circular cylinder, set $x = \cos(t)$ and $y = \sin(t)$. Next, use the second equation to solve for z, so $z = 2 - 3x + 2y = 2 - 3\cos(t) + 2\sin(t)$. Therefore

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 2 - 3\cos(t) + 2\sin(t) \rangle$$

5. a) (10 points) Consider the points $A(3,1,0)$, $B(1,0,3)$ and $C(0,3,1)$. Find the area of the triangle ABC. (Hint: If you know how to find the area of a parallelogram spanned by 2 vectors, then you should be able to solve this problem.)

The area of the parallelogram is

$$\begin{aligned} |\mathbf{AB} \times \mathbf{AC}| &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 3 \\ -3 & 2 & 1 \end{vmatrix} \\ &= \left| \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 3 \\ -3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -1 \\ -3 & 2 \end{vmatrix} \mathbf{k} \right| \\ &= |\langle -7, 7, -7 \rangle| \\ &= \sqrt{147} \end{aligned}$$

So, the area of the triangle is $\sqrt{147}/2$

b) (10 points) Suppose a particle moving in space has velocity

$$\mathbf{v}(t) = \langle \cos t, \sin 3t, e^{2t} \rangle$$

and initial position $\mathbf{r}(0) = \langle 1, 2, 0 \rangle$. Find the position vector function $\mathbf{r}(t)$.

Integrating the velocity vector, we get position:

$$\mathbf{r}(t) = \langle \sin t + c_1, -\frac{1}{3}\cos(3t) + c_2, \frac{1}{2}e^{2t} + c_3 \rangle$$

Now, we insist that when $t = 0$ is plugged into the position equation, the position must be $\langle 1, 2, 0 \rangle$:

$$\mathbf{r}(0) = \langle c_1, -\frac{1}{3} + c_2, \frac{1}{2} + c_3 \rangle = \langle 1, 2, 0 \rangle$$

Therefore $c_1 = 1$, $c_2 = \frac{7}{3}$ and $c_3 = -\frac{1}{2}$, so the position vector is

$$\mathbf{r}(t) = \langle \sin t + 1, -\frac{1}{3}\cos(3t) + \frac{7}{3}, \frac{1}{2}e^{2t} - \frac{1}{2} \rangle$$