HOMOGENIZATION FOR SOME PERIODIC AND RANDOM NONLINEAR SCHRÖDINGER MODELS

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Abstract. In the present communication, we derive homogenized equations for nonlinear Schrödinger settings with periodic as well as ergodic random potentials. Our case examples are motivated by recent experimentally accessible applications in soft-condensed matter, as well as in optical physics. Particular features of the resulting equations are compared directly with corresponding predictions of the original model and good agreement is found between the two. Higher order corrections to the leading order homogenization results are also discussed.

1. Introduction

In the past few years, there has been an explosion of interest in nonlinear Schrödinger equations in periodic media [1, 2, 3, 4]. This has been particularly fueled by the numerous experimental efforts in soft-condensed matter physics of Bose-Einstein condensates (BECs) in optical lattices [2, 3, 5, 6] and by the equally prolific experimental efforts in photonic crystal lattices in photorefractive materials [7, 8].

In the atomic/soft-condensed matter physics of BECs, optical lattice potentials are generated through the interference of counter-propagating laser beams and can be generated equally efficiently in one [2, 3, 5] as well as in two [9] and three [10] dimensions. In this framework, recent experimental findings include the formation of solitary waves (so-called "gap solitons") [11], the observation of the modulational instability of quasi-uniform states [12], the observation of Landau-Zener tunneling between the different bands of the periodic potential [13] and the observation of parametric resonances for time-dependent optical lattices [14], among many others. In this case, the fundamental nonlinear model that has been very successful in capturing this phenomenology is the nonlinear Schrödinger equation (NLS) with a periodic potential in 1, 2 and even 3 dimensions (depending on the experimental framework).

On the other hand, the suggestion for the generation of optically-induced lattices in photorefractive materials (such as SBN) [15], has led to a large volume of literature on the formation of nonlinear waves and coherent structures in this periodic nonlinear setting. Among the prominent examples are the observation of regular discrete solitons, dipole solitons, soliton-trains, soliton-necklaces and vector solitons [7], while, last year, two groups were independently able to experimentally produce robust discrete vortex states [8]. In this setting, however, the prototypical model that has been used to capture the relevant phenomenology encompasses the photorefractive nonlinearity proposed in [16]. Another twist in this case is that the periodic potential is an effective potential that arises due to the interaction of the two polarizations of light in the crystal; the first is the so-called extraordinary, weak probe beam that is nonlinear in its evolution, while the second is the ordinary, strong

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beam which is linear in its evolution and forms a standing wave. The latter is "perceived" by the probe beam as an external periodic potential [15, 16, 17].

Furthermore, very recent work in ultracold atom systems and BECs has allowed to impose not merely periodic but also random potentials [18, 19] and to observe various features such as collective oscillations of the condensates in these random landscapes. Another direction of very controllable (and potentially random) variation is the experimental ability to control the scattering length of inter-particle interactions in the condensates, via the so-called Feshbach resonance technique [20, 21]. The latter empowers a periodic or even random variation of the strength of the NLS nonlinearity prefactor, a feature that can be used in a variety of ways, including the avoiding of wave collapse in higher dimensions [22].

In view of the above developments, the aim of the present work is to develop an averaging/homogenization approach that will highlight the leading effective dynamics (in some cases even explicitly) in the context of such periodic and random potentials motivated from these recent studies. We will, in particular, derive averaged equations to leading order, providing as well, the second order corrections and testing the effectiveness of such predictions by comparing the solitary wave features of the resulting effective equations with the original ones. We will also touch upon random settings, providing a homogenerized theory for a potentially random, Feshbach-resonance induced, tuning of the nonlinearity. We will see that the latter will lead to a number of interesting directions as regards the numerical implementation of both the original model, as well as the effective equation.

Our presentation is structured as follows. We first present the deterministic homogenization results for a general external potential, along with the specific applications in the photorefractive and BEC context. We then examine the case of random variation of the scattering length, motivated by the freedom in the variation of the nonlinearity prefactor available in BECs. Finally, section 4, we summarize our findings and present our conclusions, as well as a number of directions of interest for future studies.

2. Deterministic Homogenization With Spatially Periodic Coefficients

We are interested in the large n behavior of the following two NLS equations in 1-D

$$i\frac{\partial}{\partial t}u(t,x) = -\Delta_{xx}u(t,x) + V_0\sin^2(nx)u(t,x) + |u(t,x)|^2u(t,x). \tag{2.1}$$

and

$$i\frac{\partial}{\partial t}u(t,x) = -\Delta_{xx}u(t,x) + \frac{1}{1+a_0\sin^2(nx)+|u(t,x)|^2}u(t,x) + V(x)u(t,x), \tag{2.2}$$

The first one of these models arises in the study of BECs in optical lattices [2, 3], while the second one is relevant to photonic crystal lattices in photorefractives [15, 7, 8, 16].

The above equations are special cases of

$$i\frac{\partial}{\partial t}u(t,x) = -\Delta_{xx}u(t,x) + F(x,nx,|u(t,x)|^2)u(t,x). \tag{2.3}$$

by taking

$$F(x,y,r) = V_0\sin^2(y) + \sigma r, \quad \sigma = \pm 1,$$

and

$$F(x,y,r) = (1+a_0\sin^2(y)+r)^{-1} + V(x), \quad r \geq 0,$$

respectively.

Throughout, we assume that F is sufficiently smooth in all variables and is periodic in y with periodicity π. These are satisfied by the examples we have in mind.

Let

$$A_n g(x) = -i\Big(- \Delta_{xx} g(x) + F(x, nx, |g(x)|^2) g(x)\Big).$$

Equation (2.3) can be written in the usual evolution equation form

$$\partial_t u_n = A_n u_n.$$

Loosely speaking, we are interested in identifying the effective dynamics governing the first two order terms in asymptotic expansion of u_n in the form

$$u_n(t,x) = u_0(t,x) + n^{-1} v_0(t,x) + o(n^{-1}). \tag{2.4}$$

Our result shows that u_0 is described by (2.7) and v_0 by (2.10).

2.1. **Leading order homogenization.** Suppose the limiting u_0 satisfy

$$\partial_t u_0(t,x) = A u_0(t,x).$$

We identify the operator A below.

Informally apply the results in semigroup theory (see e.g. Kurtz [23]), A should be the operator graph limit of the A_ns, in the sense that for each test function f in the domain of A, there should exist f_ns in the domain of A_n, such that $f_n \to f$ and $A_n f_n \to Af$. In the specific context here, we claim that it is sufficient to choose f_n according to the form

$$f_n(x) = f(x) + n^{-2} g(x, nx).$$

We verify this claim and identify the g next.

First, denote

$$g_{11}(x,y) = \Delta_{xx}(x,y), \quad g_{12}(x,y) = \nabla_x \cdot \nabla_y g(x,y), \quad g_{22}(x,y) = \Delta_{yy} g(x,y),$$

then

$$\Delta_{xx} f_n(x) = \Delta f(x) + n^{-2} g_{11}(x, nx) + 2n^{-1} g_{12}(x, nx) + g_{22}(x, nx)$$

so

$$A_n f_n(x) = -i\Big(- \Delta f(x) - g_{22}(x, nx) + F(x, nx, |f(x)|^2) f(x)\Big) + O(n^{-1})$$

Viewing nx as a new variable y, we see that the above expresses a separation of variable at two different scales. In order for $A_n f_n(x)$ to be asymptotically independent of n (hence having a limit), we need to choose $g(x,y) = g_f(x,y)$ so that

$$-\Delta_{xx} f(x) - \Delta_{yy} g(x,y) + F(x, y, |f(x)|^2) f(x) = h(x) \tag{2.5}$$

for some $h = h_f$ independent of y. The resulting limit of $A_n f_n$ then becomes

$$Af(x) = -ih(x).$$

Let $\mathcal{O} = [0, \pi)$ with periodic boundary. By Fredholm alternative, then (2.5), as an equation in y with x fixed, is solvable provided

$$\int_{\mathcal{O}} (-\Delta_{xx} f(x) + F(x, y, |f(x)|^2) f(x) - h(x)) \mu(dy) = 0,$$

where μ is the measure such that: letting $T(t)$ be the semigroup generated by Δ, then

$$\lim_{t \to +\infty} t^{-1} \int_0^t T(s) f(x) ds = \int_{\mathcal{O}} f(y) \mu(dy), \quad \forall f \in C_b(\mathcal{O}), \forall x \in \mathcal{O}. \tag{2.6}$$

With the periodic boundary condition on $\mathcal{O}$, $\mu(dy) = (2\pi)^{-1}dy$. Hence (2.6) implies

$$h(x) = -\Delta_{xx} f(x) + \bar{F}(x, |f(x)|^2) f(x)$$

where

$$\bar{F}(x, r) = \int_{y \in \mathcal{O}} F(x, y, r) \mu(dy).$$

In conclusion, the limiting u should satisfy

$$i \frac{\partial}{\partial t} u_0(t, x) = -\Delta_{xx} u_0(t, x) + \bar{F}(x, |u_0(t, x)|^2) u_0(t, x). \tag{2.7}$$

2.2. **Order-n^{-1} correction.** The above analysis suggests that we can approximate u_n by u_0. Below, we analyze the error of such approximation. Let

$$v_n(t, x) = n(u_n(t, x) - u_0(t, x)).$$

We show that $v_n \to v_0$ where v_0 is described by (2.10).

We start by observing that the evolution of (v_n, u_0) form a closed system

$$\begin{aligned} i \frac{\partial}{\partial t} v_n(t, x) &= -\Delta v_n(t, x) + n\Big(F(x, nx, |u_0(t, x) + n^{-1} v_n(t, x)|^2)(u_0(t, x) + n^{-1} v_n(t, x)) \\ &\qquad - \bar{F}(x, |u_0(t, x)|^2) u_0(t, x) \Big) \\ i \frac{\partial}{\partial t} u_0(t, x) &= -\Delta u_0(t, x) + \bar{F}(x, |u_0(t, x)|^2) u_0(t, x). \end{aligned}$$

Thinking $(v_n(t), u_0(t))_{t \geq 0}$ in terms of a solution semigroup on the space of two square integrable complex functions, its generator is given by

$$\begin{aligned} \Big(B_n(v, u) \Big)(x) &= -i\Big(-\Delta v(x) + nF(x, nx, |u(x) + n^{-1} v(x)|^2)(u(x) + n^{-1} v(x)) \\ &\qquad - n\bar{F}(x, |u(x)|^2) u(x), -\Delta u + \bar{F}(x, u(x)) u(x) \Big) \\ &= -i\Big(-\Delta v + F(x, nx, |u(x)|^2) v(x) \\ &\qquad + F_3(x, nx, |u(x)|^2)(u(x) v^*(x) + v(x) u^*(x)) u(x) \\ &\qquad + n(F(x, nx, |u(x)|^2) - \bar{F}(x, |u(x)|^2) u(x)), \\ &\qquad -\Delta u + \bar{F}(x, u(x)) \Big) + O(n^{-1}), \end{aligned} \tag{2.8}$$

where

$$F_3(x, y, r) = \frac{\partial}{\partial r} F(x, y, r)$$

and v^* is the complex conjugate for v.

Again, we expect the limit of B_n should be given in the operator graph convergence sense, and denote the limit operator B:

$$\exists (v_n, u_n) \to (v, u), \quad \text{such that } B_n(v_n, u_n) \to B(v, u).$$

We claim that it is good enough to consider v_n, u_ns of the following form with f, g to be identified later

$$v_n(x) = v(x) + n^{-1} f(x, nx) + n^{-2} g(x, nx), \quad u_n(x) = u(x).$$

Noting

$$\begin{aligned}\Delta_{xx}v_n(x) &= \Delta_{xx}v(x) + n^{-1}(f_{11}(x,nx) + 2nf_{12}(x,nx) + n^2 f_{22}(x,nx)) \\ &\quad + n^{-2}(g_{11}(x,nx) + 2ng_{12}(x,nx) + n^2 g_{22}(x,nx)), \\ &= nf_{22}(x,nx) + \Delta_{xx}v(x) + 2f_{12}(x,nx) + g_{22}(x,nx) + O(n^{-1}),\end{aligned}$$

the first component in $B_n(v_n, u_n)$ (see (2.8)) becomes

$$\begin{aligned}-i \quad \Big(&- \Delta_{xx}v(x) - 2f_{12}(x,nx) - g_{22}(x,nx) + F(x,nx,|u(x)|^2)v(x) \\ &+ F_3(x,nx,|u|^2)(u^2(x)v^*(x) + |u(x)|^2 v(x)) \\ &+ n(-f_{22}(x,nx) + (F(x,nx,|u(x)|^2) - \bar{F}(x,|u(x)|^2))u(x))\Big) + O(n^{-1}).\end{aligned}$$

As in last section, we would like to select f and g approporiately depending on v, u so that the above term is asympototically independent of nx. To annihilate the order-n term, we need to solve

$$\Delta_{yy}f(x,y) = F(x,y,|u(x)|^2)u(x) - \bar{F}(x,|u(x)|^2)u(x)$$

for all $x, u(x)$ fixed. By definition of $\bar{F}$, this equation has a solution which is given by

$$\begin{aligned}f(x,y) &= (\Delta)^{-1}(F(x,\cdot,|u(x)|^2) - \bar{F}(x,|u(x)|^2))(y)u(x) \\ &= \{\int_0^\infty T(t)(F(x,\cdot,|u(x)|^2) - \bar{F}(x,|u(x)|^2))dt\}(y)u(x),\end{aligned}$$

where $T(t)$ is the semigroup generated by Δ. Or equivalently, let $W(t)$ be a Brownian motion on the torus $\mathcal{O}$, then

$$f(x,y) = u(x)E[\int_0^\infty (F(x,W(t),|u(x)|^2) - \bar{F}(x,|u(x)|^2))dt|W(0) = y].$$

For the order-1 term to be independent of nx asymptotically, we need to solve

$$\begin{aligned}-\Delta_{xx}v(x) - 2\nabla_y\nabla_x f(x,y) - \Delta_{yy}g(x,y) + F(x,y,|u(x)|^2)v(x) \\ + F_3(x,y,|u(x)|^2)(u^2(x)v^*(x) + |u(x)|^2 v(x)) = h(x)\end{aligned}$$

for some h independent of y. This equation for y will have a solution if only if (again, by the Fredholm alternative)

$$\begin{aligned}\int_{\mathcal{O}}(&-\Delta_{xx}v(x) - 2\nabla_y\nabla_x f(x,y) + F(x,y,|u(x)|^2)v(x) \\ &+ F_3(x,y,|u(x)|^2)(u^2(x)v^*(x) + |u(x)|^2 v(x)) - h(x))\mu(dy) = 0 \qquad (2.9)\end{aligned}$$

where μ is the uniform measure on $\mathcal{O}$, the ergodic measure for the semigroup $T(t)$. We notice that

$$\int_{\mathcal{O}}(\nabla_y\nabla_x f(x,y))1dy = 0$$

therefore (2.9) imples

$$h(x) = -\Delta_{xx}v(x) + \bar{F}(x,|u(x)|^2)v(x) + \bar{F}_3(x,|u(x)|^2)(u^2(x)v^*(x) + |u(x)|^2 v(x))$$

where

$$\bar{F}_3(x,z) = \int_{y\in\mathcal{O}} F_3(x,y,z)\mu(dy).$$

In conclusion, $v_0(t,x)$ (the limit of v_n) should satisfy a linear Schrödinger equation with an effective potential which depends on u_0 in (2.7):

$$i\frac{\partial}{\partial t}v_0 = -\Delta_{xx}v_0 + \bar{F}(x,|u_0|^2)v_0 + \bar{F}_3(x,|u|^2)(u^2v_0^* + |u|^2v_0) \tag{2.10}$$

2.3. **Examples and numerical results.**

Example 2.1 (Equation (2.1))**.** In this case,

$$F(x,y,r) = V_0\sin^2(y) + \sigma r,$$

(where $\sigma = \pm 1$) we have

$$\bar{F}(x,r) = (\pi)^{-1}\int_{y\in(0,\pi]} F(x,y,r)dy = \frac{1}{2}\pi V_0 + \sigma r,$$

and

$$F_3(x,y,r) = \frac{\partial}{\partial r}F(x,y,r) = \sigma$$

and

$$\bar{F}_3(x,r) = (\pi)^{-1}\int_{y\in(0,\pi]} F_3(x,y,r)dy = \sigma.$$

So (2.7) reduces to

$$i\partial_t u_0 = -\Delta_{xx}u_0 + \frac{1}{2}V_0u_0 + \sigma|u_0|^2u_0,$$

and (2.10) reduces to

$$i\partial_t v_0 = -\Delta_{xx}v_0 + \frac{1}{2}V_0v_0 + \sigma|u_0|^2v_0 + \sigma(u_0^2v_0^* + |u_0|^2v_0).$$

We now compare these predictions to direct numerical results. In this context, it is interesting to note that the equation for u_0 is nothing but a regular NLS equation with a renormalized frequency. I.e., for $\tilde{u}_0 = u_0\exp(-iV_0t/2)$, the equation for $\tilde{u}_0$ is nothing but a regular NLS. If we consider more specifically the case of $\sigma = -1$, a standard solution for $\tilde{u}_0$ comes in the form of solitons

$$\tilde{u}_0 = \sqrt{2\Lambda}\text{sech}\left(\sqrt{\Lambda}(x-\xi)\right)\exp(i\Lambda t), \tag{2.11}$$

where Λ is the (arbitrary) frequency of the solution and ξ its arbitrary center location. Solutions for u_0 can be obtained immediately thereafter and directly compared to the solution of the full problem in the presence of the periodic potential, for various values of n, so that the range of validity of the averaging theory can be elucidated. This is done in Fig. 1, where the norms of the full (dashed) and averaged (solid) solutions are compared and their profiles are also explicitly given for a range of values of n. It is clear that for large n the theory is practically exact and the rescaling of the frequency fully captures the effect of the periodic external potential. However, as n decreases, the two profiles start deviating, especially as the size of the soliton becomes comparable to the size of the lattice spacing between the potential wells. The corrections could, in principle, be captured by analyzing the linear correction problem (which is explicitly solvable due to the complete integrability of the NLS equation [24]). However, given the effectiveness of the approximation (essentially even down to $n = 1$) we will not proceed in that direction here.

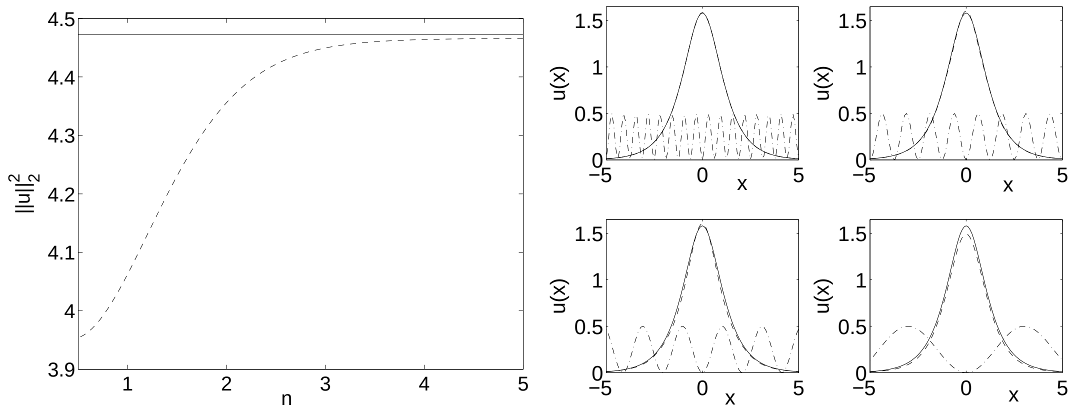

FIGURE 1. The left panel shows the norm of the averaged exact solution for u_0 (solid line) versus the one of the full problem for different values of n (dashed line). The right panel compares the two solution profiles (averaged by solid line, versus full in dashed line) for different n's, namely for $n = 5$ (top left), $n = 2.5$ (top right), $n = 1.5$ (bottom left) and $n = 0.5$ (bottom right). The corresponding potentials are shown by dash-dotted lines in all cases.

Example 2.2 (Equation (2.2))**.** In the case where

$$F(x,y,r) = (1 + a_0 \sin^2(y) + r)^{-1},$$

we have

$$\bar{F}(x,r) = (\pi)^{-1} \int_{y\in(0,\pi]} F(x,y,r)dy = \frac{1}{\sqrt{(1+r)(1+a_0+r)}},$$

and

$$F_3(x,y,r) = \frac{\partial}{\partial r} F(x,y,r) = -(1 + a_0 \sin^2(y) + r)^{-2}$$

and

$$\bar{F}_3(x,r) = (\pi)^{-1} \int_{y\in(0,\pi]} F_3(x,y,r)dy = -\frac{1 + a_0/2 + r}{(1+r)^{3/2}\sqrt{1+a_0+r}}.$$

So (2.7) reduces to

$$i\partial_t u_0 = -\Delta_{xx} u_0 + \frac{1}{\sqrt{(1+|u_0|^2)(1+a_0+|u_0|^2)}} u_0, \tag{2.12}$$

and (2.10) reduces to

$$i\partial_t v_0 = -\Delta_{xx} v_0 + \frac{1}{\sqrt{(1+|u_0|^2)(1+a_0+|u_0|^2)}} v_0 - \frac{1+a_0/2+|u_0|^2}{(1+|u_0|^2)^{3/2}\sqrt{1+a_0+|u_0|^2}} (u_0^2 v_0^* + |u_0|^2 v_0).$$

Here also, we can compare the relevant homogenization prediction with the corresponding full numerical result. However, in this case, the homogenized equation cannot be solved analytically (to the best of our knowledge) for the solitary wave profile and the relevant result needs to be obtained numerically as well. This is to some extent a drawback of the application of the method in the present case, however the homogeneous problem of Eq. (2.12) is considerably easier to solve than the full problem of Eq. (2.2), as there is no need for the careful numerical resolution of the lattice details. In addition, as we see in Fig. 2,

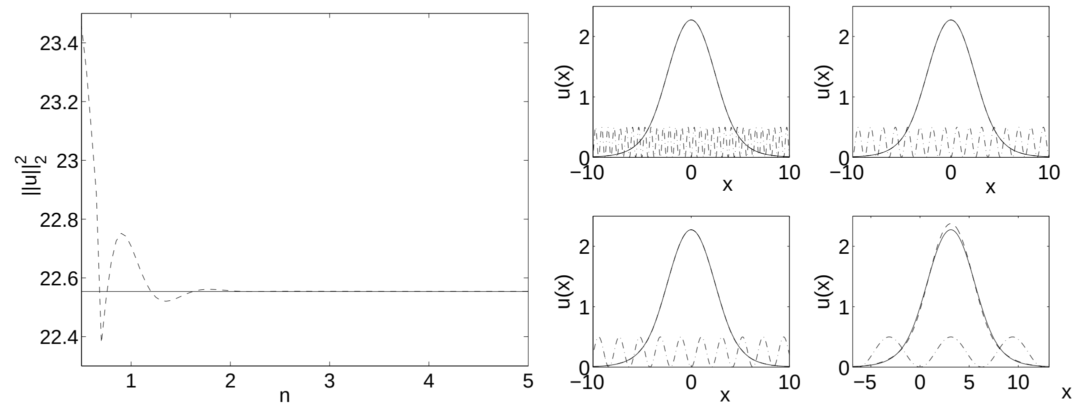

FIGURE 2. The panels are identical as those of Fig. 1, but for the photorefractive nonlinear PDE of Eq. (2.2). Once again, $V_0 = 0.5$ is used.

the result of the homogenization theory remains valid practically down to the $n = 1$ in this setting as well.

3. The Case of Stochastic Homogenization For Random Nonlinearity Coefficients

Our motivating example in this case is

$$i\frac{\partial}{\partial t}u_n(t,x) = -\Delta_{xx}u_n(t,x) + V(x)u_n(t,x) + \alpha_n(Y(n^2t))|u_n(t,x)|^2u_n(t,x). \tag{3.1}$$

In the above, $\alpha_n : S \to R$ is a function controlling the strength of particle-particle interaction (in BECs). It is modulated by an external environment factor modeled by a stochastic process $Y(n^2\cdot)$ on a compact metric space S. We are interested in the asymptotic behavior of the solution when $n \to +\infty$; that is, when the random environment becomes highly oscilating. We will rescale α_n in a way to be explained below. A good example to keep in mind is that α_n switches between -1 and $+1$, depending on the value of $Y(n^2t)$.

3.1. **Structural assumptions.** By taking $F_n(x,y,r) = V(x) + \alpha_n(y)r$, the above model can be written in a general form (where F_n is real valued)

$$i\frac{\partial}{\partial t}u_n(t,x) = -\Delta_{xx}u_n(t,x) + F_n(x, Y(n^2t), |u_n(t,x)|^2)u_n(t,x). \tag{3.2}$$

There are several possible scalings which will give meaningful limits. We consider only the case

$$\alpha_n(y) = \alpha_0(y) + n\alpha_1(y),$$

and

$$F_n(x,y,r) = G(x,y,r) + n\alpha_1(y)U(r) \tag{3.3}$$

with a centering condition

$$\int_{y\in S}\alpha_1(y)\pi_0(dy) = 0. \tag{3.4}$$

It follows that (3.1) is a special situation of the above model by taking

$$G(x,y,r) = V(x) + \alpha_0(y)r, \quad U(r) = r.$$

3.2. **Asymptotic limit.** To avoid unnecessary complex variable notations, we view (3.2) as a system of real valued PDEs. Throughout, we use

$$u(x) = v(x) + iw(x)$$

where v, w are real valued functions, to represent the complex valued solution u of (3.2). Therefore,

$$\begin{aligned} \partial_t v_n &= -\Delta_{xx} w_n + F_n(x, Y(n^2 t), v_n^2 + w_n^2) w_n \\ \partial_t w_n &= \Delta_{xx} v_n - F_n(x, Y(n^2 t), v^2 + w^2) v_n. \end{aligned} \tag{3.5}$$

In deriving the deterministic homogenization problems in earlier sections, we made essential use of semigroup/generator-graph convergence approach. In this probabilistic example, we modify the approach to incorporate the randomness. We will make use of the so-called *martingale problem method* due to Stroock and Varadhan [26]. See Ethier and Kurtz [25] for generalizations of the method allowing the situation here. We stress that our derivation is only semi-formal. A rigorous derivation would also involve additionally proving tightness (i.e. relative compactness) of the sequence of solutions $\{u_n : n = 1, 2, \ldots\}$, and showing uniqueness of the martingale problem for A (i.e. uniqueness of the stochastic PDE (3.7)).

For smooth test functions $f(v, w)$, we define the first and second variational derivative according to Taylor expansion

$$\begin{aligned} & f(v + \bar{v}, w + \bar{w}) - f(v, w) \\ = \; & \langle \frac{\delta f}{\delta v}, \bar{v} \rangle + \langle \frac{\delta f}{\delta w}, \bar{w} \rangle \\ & \quad + \frac{1}{2} \int \int \Big(\frac{\delta^2 f}{\delta v^2}(x_1, x_2) \bar{v}(x_1) \bar{v}(x_2) + \frac{\delta^2 f}{\delta w^2}(x_1, x_2) \bar{w}(x_1) \bar{w}(x_2) + 2 \frac{\delta^2 f}{\delta v \delta w} \bar{w}(x_1) \bar{v}(x_2) \Big) dx_1 dx_2 \\ & \qquad + o(\|v\|_{L^2}^2 + \|w\|_{L^2}^2) \end{aligned}$$

Let real valued functions $\varphi_k \in C_c^\infty$ and real valued function $\psi \in C^1(R^{2m})$. We first consider test functions of the form

$$f(v, w) = \psi(\langle \varphi_1, v \rangle, \ldots, \langle \varphi_m, v \rangle; \langle \varphi_{m+1}, w \rangle), \ldots, \langle \varphi_{2m}, w \rangle), \tag{3.6}$$

It follows then

$$\begin{aligned} & f(v_n(t), w_n(t)) - f(v_n(s), w_n(s)) \\ = \; & \int_s^t \Big(\sum_{k=1}^m \partial_k \psi(\langle \varphi_1, v(r) \rangle, \ldots, \langle \varphi_m, v(r) \rangle; \langle \varphi_{m+1}, w(r) \rangle), \ldots, \langle \varphi_{2m}, w(r) \rangle) \\ & \qquad \cdot \langle -\Delta w(r) + F_n(\cdot, Y(n^2 r), v^2(r) + w^2(r)) w(r), \varphi_k \rangle \\ & \quad + \sum_{k=m+1}^{2m} \partial_k \psi(\langle \varphi_1, v(r) \rangle, \ldots, \langle \varphi_m, v(r) \rangle; \langle \varphi_{m+1}, w(r) \rangle), \ldots, \langle \varphi_{2m}, w(r) \rangle) \\ & \qquad \cdot \langle \Delta v(r) - F_n(\cdot, Y(n^2 r), v^2(r) + w^2(r)) v(r), \varphi_k \rangle \Big) dr \\ = \; & \int_s^t \Big(\langle -\Delta w(r) + F_n(\cdot, Y(n^2 r), v^2(r) + w^2(r)) w(r), \frac{\delta f}{\delta v} \rangle \\ & \quad + \langle \Delta v(r) - F_n(\cdot, Y(n^2 r), v^2(r) + w^2(r)) v(r), \frac{\delta f}{\delta w} \rangle \Big) dr. \end{aligned}$$

However, the evolution of (v_n, w_n) alone is not a closed system, we should include $Y(n^2\cdot)$. Therefore, our smooth test function f should also have a y variable in general. Let B be the generator for $Y(\cdot)$ (see [25]), then $(v_n, w_n, Y(n^2\cdot))$ is a Markov process with generator A_n to be determined below. For test functions of the form

$$f(v,w;y) = \psi(\langle\varphi_1, v\rangle, \ldots, \langle\varphi_m, v\rangle; \langle\varphi_{m+1}, w\rangle), \ldots, \langle\varphi_{2m}, w\rangle; y),$$

we define

$$A_n f(v,w,y) = \langle -\Delta w + F_n(\cdot, y, v^2+w^2)w, \frac{\delta f}{\delta v}\rangle + \langle \Delta v - F_n(\cdot, y, v^2+w^2)v, \frac{\delta f}{\delta w}\rangle\Big) + n^2 B_y f(v,w,y).$$

In the above, B_y (we will omit the subscript y later) means B acts on the y variable of f only. In the martingale problem formulation of Stroock and Varadhan [26],

$$f(u_n(t), Y_n(t)) - f(u_n(s), Y_n(s)) - \int_s^t A_n f(u_n(r), Y_n(r))dr = M_n^f(t) - M_n^f(s),$$

where M^f is martingale.

The generator for the Markov process $(v_n(\cdot), w_n(\cdot), Y(n^2\cdot))$ is then

$$\begin{aligned} A_n f(v,w,y) &= \langle -\Delta w + F_n(\cdot, y, v^2+w^2)w, \frac{\delta f}{\delta v}\rangle \\ &\quad + \langle \Delta v - F_n(\cdot, y, v^2+w^2)v, \frac{\delta f}{\delta w}\rangle + n^2 B_y f(v,w,y). \end{aligned}$$

Choose

$$f_n(v,w,y) = f(v,w) + n^{-1}g(v,w,y) + n^{-2}h(v,w,y)$$

where $f(v,w)$ is of the form (3.6), and g and h will be decided using concrete calculations next. Then

$$\begin{aligned} &A_n f_n(v,w,y) \\ = \;& n\Big(\alpha_1(y)\langle U(v^2+w^2)w, \frac{\delta f}{\delta v}\rangle + \alpha_1(y)\langle -U(v^2+w^2)v, \frac{\delta f}{\delta w}\rangle + Bg(v,w,y)\Big) \\ &+\Big(\langle -\Delta w + G(\cdot, y, v^2+w^2)w, \frac{\delta f}{\delta v}\rangle + \langle \Delta v - G(\cdot, y, v^2+w^2)v, \frac{\delta f}{\delta w}\rangle \\ &+\alpha_1(y)\langle U(v^2+w^2)w, \frac{\delta g}{\delta v}(\cdot; y)\rangle + \alpha_1(y)\langle -U(v^2+w^2)v, \frac{\delta g}{\delta w}(\cdot; y)\rangle + Bh(v,w,y)\Big) + O(n^{-1}). \end{aligned}$$

As in the previous examples, our task is to derive a limit of $A_n f_n$, independent of y, by selecting g, h carefully.

A word of caution is necessary, the form of g and h will indeed not be of the form (3.6) any more. However, given the density of the class of test functions of the form (3.6), we expect A_n should not change its form (expressed in terms of variational derivatives) even if we replace f by the f_n. This can also be verified directly.

In order for $A_n f_n$ not to blow up as $n \to +\infty$, we have to choose g so that

$$\alpha_1(y)\langle U(v^2+w^2)w, \frac{\delta f}{\delta v}\rangle + \alpha_1(y)\langle -U(v^2+w^2)v, \frac{\delta f}{\delta w}\rangle + Bg(v,w,y) = 0.$$

Defining

$$\eta(y, dz) = \int_0^\infty (P(Y(t) \in dz | Y(0) = y) - \pi_0(dz))dt.$$

As far as Y is sufficiently ergodic (for instance, the speed of convergence to the unique equilibrium measure π_0 is of order $O(t^{-2})$ at large time), then the above quantity exists and is well defined. Noting (3.4), the following is a finite, real valued function

$$(P_\eta\alpha_1)(y) \equiv B^{-1}(-\alpha_1)(y) = \int_{z\in S} \alpha_1(z)\eta(y,dz).$$

Recalling that f is of the form (3.6), we therefore have

$$g(v,w,y) = P_\eta\alpha_1(y)\Big(\langle U(v^2+w^2)w, \frac{\delta f}{\delta v}\rangle - \langle U(v^2+w^2)v, \frac{\delta f}{\delta w}\rangle\Big).$$

Consequently,

$$\begin{aligned}
&\frac{\delta g}{\delta v}(x;y) \\
= \ & P_\eta\alpha_1(y)\Big\{ \sum_{l,k=1}^{m} \partial^2_{lk}\psi(\langle\varphi_1,v\rangle,\ldots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\ldots,\langle\varphi_{2m},w\rangle)\langle U(v^2+w^2)w,\varphi_k\rangle\varphi_l(x) \\
& - \sum_{k=m+1}^{2m}\sum_{l=1}^{m} \partial^2_{lk}\psi(\langle\varphi_1,v\rangle,\ldots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\ldots,\langle\varphi_{2m},w\rangle)\langle U(v^2+w^2)v,\varphi_k\rangle\varphi_l(x) \\
& +2U'(v^2(x)+w^2(x))v(x)w(x)\frac{\delta f}{\delta v}(x) \\
& -\Big(U(v^2(x)+w^2(x)) + 2U'(v^2(x)+w^2(x))v^2(x)\Big)\frac{\delta f}{\delta w}(x)\Big\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\delta g}{\delta w}(x;y) \\
= \ & P_\eta\alpha_1(y)\Big\{ \sum_{l=m+1}^{2m}\sum_{k=1}^{m} \partial^2_{lk}\psi(\langle\varphi_1,v\rangle,\ldots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\ldots,\langle\varphi_{2m},w\rangle)\langle U(v^2+w^2)w,\varphi_k\rangle\varphi_l(x) \\
& - \sum_{k,l=m+1}^{2m} \partial^2_{lk}\psi(\langle\varphi_1,v\rangle,\ldots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\ldots,\langle\varphi_{2m},w\rangle)\langle U(v^2+w^2)v,\varphi_k\rangle\varphi_l(x) \\
& +\Big(U(v^2(x)+w^2(x)) + 2U'(v^2(x)+w^2(x))w^2(x)\Big)\frac{\delta f}{\delta v}(x) \\
& -2U'(v^2(x)+w^2(x))v(x)w(x)\frac{\delta f}{\delta w}(x)\Big\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&A_n f_n(v,w;,y)\\
\to\ &\langle -\Delta w + G(\cdot,y,v^2+w^2)w - \alpha_1(y)P_\eta\alpha_1(y)U^2(v^2+w^2)v, \frac{\delta f}{\delta v}\rangle\\
&+\langle \Delta v - G(\cdot,y,v^2+w^2)v - \alpha_1(y)P_\eta\alpha_1(y)U^2(v^2+w^2)w, \frac{\delta f}{\delta w}\rangle\\
&+\alpha_1(y)P_\eta\alpha_1(y)\Big\{ \sum_{l,k=1}^{m} \partial^2_{kl}\psi(\langle\varphi_1,v\rangle,\dots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\dots,\langle\varphi_{2m},w\rangle)\\
&\qquad \langle U(v^2+w^2)w,\varphi_k\rangle\langle U(v^2+w^2)w,\varphi_l\rangle\\
&- \sum_{k=m+1}^{2m}\sum_{l=1}^{m} \partial^2_{kl}\psi(\langle\varphi_1,v\rangle,\dots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\dots,\langle\varphi_{2m},w\rangle)\\
&\qquad \langle U(v^2+w^2)v,\varphi_k\rangle\langle U(v^2+w^2)w,\varphi_l\rangle\\
&- \sum_{l=m+1}^{2m}\sum_{k=1}^{m} \partial^2_{kl}\psi(\langle\varphi_1,v\rangle,\dots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\dots,\langle\varphi_{2m},w\rangle)\\
&\qquad \langle U(v^2+w^2)w,\varphi_k\rangle\langle U(v^2+w^2)v,\varphi_l\rangle\\
&+ \sum_{k,l=m+1}^{2m} \partial^2_{kl}\psi(\langle\varphi_1,v\rangle,\dots,\langle\varphi_m,v\rangle;\langle\varphi_{m+1},w\rangle),\dots,\langle\varphi_{2m},w\rangle)\\
&\qquad \langle U(v^2+w^2)v,\varphi_k\rangle\langle U(v^2+w^2)v,\varphi_l\rangle\Big\}\\
&+Bh(v,w;y).
\end{aligned}$$

As in the deterministic examples, select h to annihilate the y-variable dependence above. Let

$$\bar{G}(x,r) = \int_S G(x,y,r)\pi_0(dy),$$

and

$$\sigma^2 = \int_S \alpha_1(y)P_\eta\alpha_1(y)\pi_0(dy) \geq 0.$$

Then the limit operator is

$$\begin{aligned}
Af(v,w) \;=\; &\langle -\Delta w + \bar{G}(\cdot,v^2+w^2)w - \sigma^2U^2(v^2+w^2)v, \frac{\delta f}{\delta v}\rangle\\
&+\langle \Delta v - \bar{G}(\cdot,v^2+w^2)v - \sigma^2U^2(v^2+w^2)w, \frac{\delta f}{\delta w}\rangle\\
&+\sigma^2\Big(\langle (U(v^2+w^2)w)\otimes(U(v^2+w^2)w), \frac{\delta^2 f}{\delta v^2}\rangle\\
&-\langle (U(v^2+w^2)w)\otimes(U(v^2+w^2)v) + (U(v^2+w^2)v)\otimes(U(v^2+w^2)w), \frac{\delta^2 f}{\delta v\delta w}\rangle\\
&+\langle (U(v^2+w^2)v)\otimes(U(v^2+w^2)v), \frac{\delta^2 f}{\delta w^2}\rangle\Big),
\end{aligned}$$

where for any two single variable functions $f = f(x), g = g(x)$, we denote a double variable function

$$(f \otimes g)(x, y) = f(x)g(y).$$

Let $u(t, x) = v(t, x) + iw(t, x)$. At least formally, the stochastic partial differential equation which also solves the martingale problem for A is

$$\begin{aligned} (3.7) \qquad i\partial_t u(t,x) &= -\Delta_{xx} u(t,x) + \bar{G}(x, |u(t,x)|^2) u(t,x) - i\sigma^2 U^2(|u(t,x)|^2) u(t,x) \\ &\quad + \sqrt{2}\sigma U(|u(t,x)|^2) u(t,x) \partial_t W(t), \end{aligned}$$

where $W(t)$ is a real valued standard Brownian motion.

3.3. **Conservation law.** For the case of equation (3.1),

$$G(x, y, r) = V(x) + \alpha_0(y) r, \quad U(r) = r.$$

Hence

$$\bar{G}(x, r) = V(x) + \bar{\alpha}_0 r, \qquad \text{where } \bar{\alpha}_0 = \int_S \alpha_0(y) \pi_0(dy),$$

and and stochastic PDE (3.7) reduces to

$$(3.8) \quad i\partial_t u = -\Delta_{xx} u + V(x) u + \bar{\alpha}_0 |u|^2 u - i\sigma^2 |u|^4 u + \sqrt{2}\sigma |u(t,x)|^2 u(t,x) \partial_t W(t)$$

In the above stochastic PDE,

$$-i\sigma^2 |u|^4 u$$

can be viewed as a dissipation term and

$$\sqrt{2}\sigma |u(t,x)|^2 u(t,x) \partial_t W(t)$$

can be viewed as a fluctuation term. Their combined contributions result in the conservation of power functional. We make this precise below.

Let

$$P(u) = \int_x |u(x)|^2 dx = \int u(x) u^*(x) dx.$$

At least formally, the co-quadratic variation

$$\begin{aligned} [u(\cdot, x), u^*(\cdot, x)](t) &= [(i)^{-1}\sqrt{2} \int_0^t \sigma |u(s,x)|^2 u(s,x) dW(s), (-i)^{-1}\sqrt{2} \int_0^t \sigma |u(s,x)|^2 u^*(s,x) dW(s)] \\ &= 2\sigma^2 \int_0^t |u(s,x)|^6 ds. \end{aligned}$$

By Ito's formula,

$$
\begin{aligned}
& d\Big(u(t,x)u^*(t,x)\Big) \\
= \;& u^*(t,x)du(t,x) + u(t,x)du^*(t,x) + d[u(\cdot,x),u^*(\cdot,x)](t) \\
= \;& (i)^{-1}\Big(- u^*(t,x)\Delta_{xx}u(t,x) + V(x)|u(t,x)|^2 + \bar{\alpha}_0|u(t,x)|^4 - i\sigma^2|u(t,x)|^6\Big)dt \\
& \qquad +(i)^{-1}\sqrt{2}\sigma|u(t,x)|^4 dW(t) \\
& \quad +(-i)^{-1}\Big(- u(t,x)\Delta_{xx}u^*(t,x) + V(x)|u(t,x)|^2 + \bar{\alpha}_0|u(t,x)|^4 + i\sigma^2|u(t,x)|^6\Big)dt \\
& \qquad +(-i)^{-1}\sqrt{2}\sigma|u(t,x)|^4 dW(t) \\
& \quad +2\sigma^2|u(t,x)|^6 dt \\
= \;& (i)^{-1}\Big(- u^*(t,x)\Delta_{xx}u(t,x) + u(t,x)\Delta_{xx}u^*(t,x)\Big)dt.
\end{aligned}
$$

Therefore

$$
\int_x |u(t,x)|^2 dx = \text{constant}, \quad \forall t \geq 0, \qquad \text{a.s.}
$$

3.4. **Large deviation result and phase transition.** The effective stochastic dynamics (3.7) displays noise activited phase transitions (induced by the Brownian motion term). In the small noise limit (i.e. $\sigma \to 0+$), the difficulty of such transitions can be characterized using the theory of large deviations of the Freidlin-Wentzell type [27]. For instance, in the special case of (3.8), at least formally, we can roughly estimate the probability on path space

$$
P(u(\cdot)\in A) \sim \exp\{-\sigma^{-1}\inf_{u\in A} I(u(\cdot))\}, \qquad \text{as } \sigma \to 0+,
$$

where

$$
\begin{aligned}
I(u(\cdot)) \;=\; & \frac{1}{2}\inf\{\int_0^\infty |p(s)|^2 ds : p = p(t) \text{ satisfying} \\
& \qquad i\partial_t u = -\Delta_{xx}u + V(x)u + \bar{\alpha}_0|u|^2 u + \sqrt{2}|u(t,x)|^2 u(t,x)\partial_t p(t)\}.
\end{aligned}
$$

In the above, $p(t)$ is a real valued function in t only (no x dependence). p can be viewed as an effective control variable (given by the noise) that steer the dynamics to yield u.

Many questions regarding phase transition for the system can be quantified by I. For example, let

$$
V(t;u,v) = \inf\{I(u(\cdot)) : u(0)=u, u(t)=v\}.
$$

Then by the so called contraction principle in probabilistic large deviation theory,

$$
P(u(t)\in B | u(0)=u) \sim \exp\{-\sigma^{-1}\inf_{v\in B} V(t,u,v)\},
$$

in the small σ limit. V can be viewed as defining a quasi-potential. As in classical mechanics, the time evolution of V is characterized by a first order Hamilton-Jacobi PDE. The state space, however, is infinite dimensional (complex function valued).

3.5. **An example.** We now consider a simple special case of Y, where it is a two state Markov chain on $S = \{-1, 1\}$: $Y(t) = Y(0)(-1)^{N(t)}$ where $N(t)$ is a Poisson process with intensity $\lambda > 0$. Then

$$\pi_0(dy) = \frac{1}{2}\Big(\delta_{-1}(dy) + \delta_1(dy)\Big).$$

Using the representation

$$\eta(y, dz) = \int_0^\infty (P(Y(t) \in dz | Y(0) = y)dt,$$

and the fact that

$$P(N(t) = \text{even}) = \frac{1}{2}(1 + e^{-2\lambda t}),$$

we have

$$\begin{aligned}
\eta(1,1) &= \int_0^\infty (P(N(t) = \text{even}) - \frac{1}{2})dt = (2\lambda)^{-1} \\
\eta(1,-1) &= \int_0^\infty (P(N(t) = \text{odd}) - \frac{1}{2})dt = -(2\lambda)^{-1} \\
\eta(-1,1) &= \int_0^\infty (P(N(t) = \text{odd}) - \frac{1}{2})dt = -(2\lambda)^{-1} \\
\eta(-1,-1) &= \int_0^\infty (P(N(t) = \text{even}) - \frac{1}{2})dt = (2\lambda)^{-1}.
\end{aligned}$$

The centering condition (3.4) now requires that

$$\alpha_1(-1) + \alpha_1(1) = 0.$$

Plugging in the above explicit expression of η and π, we can identify the parameters σ and $\bar{\alpha}_0$ in (3.8)

$$\bar{\alpha}_0 = \frac{1}{2}(\alpha_0(-1) + \alpha_0(1)), \qquad \sigma^2 = \frac{1}{2\lambda}(\alpha_1(1) - \alpha_1(-1))^2.$$

4. Conclusion

In this work, we have examined some periodic and random coefficient models of the nonlinear Schrödinger class motivated by recently developed applications both in nonlinear optics and in soft condensed matter physics. In the deterministic cases, after developing the general homogenization theory for the leading order, we also found the relevant equation for the order n^{-1} correction, where n is the parameter that characterizes the fast variable (nx) of the periodic potential. In these cases, we found excellent agreement, in the two specific examples considered, between the prediction of the effective theory and the solution of the original problem, even considerably beyond the limit where the homogenization is expected to be relevant.

We also studied a NLS model with random nonlinear coefficients, deriving an effective stochastic partial differential equation for its dynamics. We showed that this equation has dissipation and fluctuation terms that conspire to preserve the ensemble average versions of the deterministic conservation laws. We also gave a specific example of the calculation of the coefficients of the stochastic PDE, for a two-state Markov chain jump in the nonlinearity.

In the latter case, a natural extension of the present work would consist of developing numerical integrators for the stochastic PDE, in order to monitor its solution in comparison with that of random coefficient model. Our preliminary computations indicate that this is a non-trivial task that only high order schemes may be able to tackle, especially because of the Brownian motion term of the equation. The low order schemes that we tried (first and second order "standard" schemes) were not efficient in that respect. This is an interesting direction and we will pursued it elsewhere.

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