Math 461 – Fall 2004
Problems due November 29, 2004

1) Consider the system of polynomial equations:

$$z^2y + z^2 = 0$$
$$x^3y + x + y = -1$$
$$z + x^2 + y^3 = 0$$

Does it have finitely many solutions? What's your best bound for the number of complex solutions? What's your best bound for the number of rational solutions?

2) Let $I \subset \mathbb{Q}[x, y, z]$ be the ideal

$$I = \langle x^4y^2 + z^2 - 4xy^3z - 2y^5z, x^2 + 2xy^2 + y^4\rangle$$

Let $f = yz - x^3$.
a) Does $f \in \sqrt{I}$?
b) If your answer to **a)** is yes, find the smallest $m \in \mathbb{N}$ such that $f^m \in I$.

3) Let $f = x^3 - 2x$ and $g = y^4 + 2$.
a) Prove that for every monomial order:

$$LM(f) = x^3 \ ; \quad LM(g) = y^4 \, .$$

b) Prove that f, g are a Gröbner basis for every monomial order. **Hint:** At some point you will have two consider two possible cases.

4) **a)** Let $I, J \subset k[x_1, \ldots, x_n]$ be ideals. Show that $\sqrt{I} = \sqrt{J}$ if and only if $I \subset \sqrt{J}$ and $J \subset \sqrt{I}$.
b) Consider the following two ideals in $\mathbb{C}[x, y, z]$.

$$I = \langle x^2z^2 + x^3, xz^4 + 2x^2z^2 + x^3, y^2z - 2yz^2 + z^3, x^2y + y^3\rangle$$

$$J = \langle xz^2 + x^2, yz^2 - z^3, x^2y - x^2z, y^4 - x^3, x^4z - x^3z, z^6 + x^4, x^5 - x^4\rangle$$

Is $I = J$? Is $\sqrt{I} = \sqrt{J}$?

5) Prove that the function

$$f(x, y, z) \ = \ (x^2 + y^2)(x^2 + y^2 - 1)z + z^3 + x + y$$

has no real critical points. That is, points in $\mathbb{R}^3$ where all three partial derivatives vanish simultaneously.

6) A polynomial

$$f = \sum_{i=1}^{n} a_i x^{\alpha(i)} \in k[x_1, \ldots, x_n]$$

is said to be *homogeneous* of degree d if $|\alpha(i)| = d$ for all $i = 1, \ldots, n$. That is, if all monomials in f have the same total degree d.

An ideal $I \subset k[x_1, \ldots, x_n]$ is called *homogeneous* if it has a basis of homogeneous polynomials (not necessarily of the same degree).
Prove that if I is a homogeneous ideal and $\prec$ is a monomial order, then the reduced Gröbner basis of I consists of homogeneous polynomials. Illustrate this problem by defining a homogeneous ideal in $\mathbb{Q}[x, y, z]$ with four homogeneous generators, each containing at least two terms, of total degrees $3, 4, 2, 3$ respectively and computing its Gröbner basis with respect to *lex*, *grlex* , and *grevlex*.