Hypertoric varieties and Gale duality

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Outline

Polarized arrangements

Definition

A *polarized hyperplane arrangement* is given by a triple (V, η, ξ)

- V is a vector subspace of $\mathbb{R}^n$, not contained in any coordinate subspace
- $\eta \in \mathbb{R}^n/V$
- $\xi \in V^*$.

These data give rise to a hyperplane arrangement in $V + \eta$ with hyperplanes

$$H_i := \{x \in V + \eta \mid x_i = 0\},$$

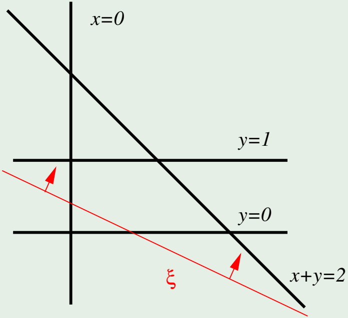
and a linear functional on $V + \eta$.

Assume η and ξ are *generic*. Then the arrangement is *simple* and the covector ξ is not constant on any positive-dimensional flat.

Example

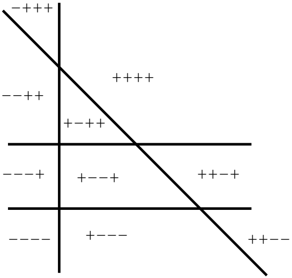
$$V = \mathrm{im} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \eta = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\xi = [\ 1 \quad 0 \quad 2 \quad 0\ ]$$



Chambers

A chamber Δ_α is given by changing the equations of the hyperplanes to inequalities, using a sign vector $\alpha \in \{+,-\}^n$



We use terminology from linear programming:

- If the chamber is nonempty, call α feasible.

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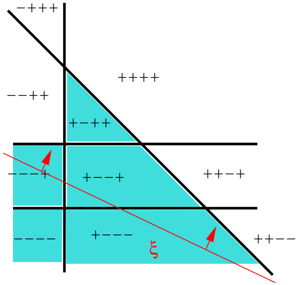


We use terminology from linear programming:

- If the chamber is nonempty, call α feasible.
- If $\xi(\Delta_\alpha)$ is bounded above, call α bounded.

Gale duality

Definition

The Gale dual of a polarized arrangement (V, η, ξ) is $(V^{\perp}, -\xi, -\eta)$, where $\mathbb{R}^n$ is identified with $(\mathbb{R}^n)^*$ in the standard way.

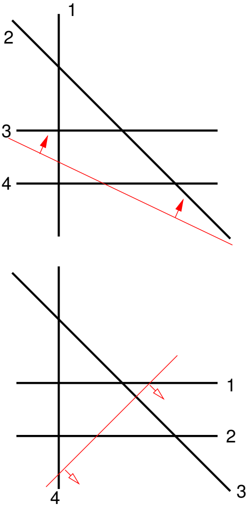
This turns an arrangment of n hyperplanes in k-space into one with n hyperplanes in $(n-k)$-space. Note the way the parameters ξ and η are interchanged.

Example of Gale duality

$$\left(\operatorname{im} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}^T \right)$$

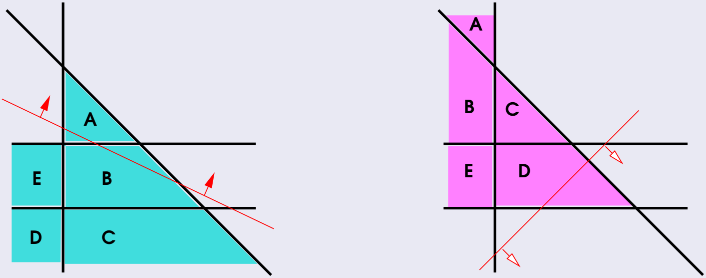
$$\Downarrow$$

$$\left(\operatorname{im} \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}^T \right)$$



Gale duality is LP duality

Gale dual polarized arrangements give dual linear programs for any sign vector α. This implies that the set of bounded feasible chambers of the two arrangements are in natural bijection. The relation "chambers α and β share a codimension 1 face" is also the same.



Hypertoric varieties

Take (V, η, ξ) with V defined/$\mathbb{Q}$ and η, ξ integral and generic.
$\rightsquigarrow$ subtorus $T_{V^\perp} \subset (\mathbb{C}^*)^n$ with Lie $T_{V^\perp} = V^\perp \otimes_{\mathbb{Q}} \mathbb{C}$.

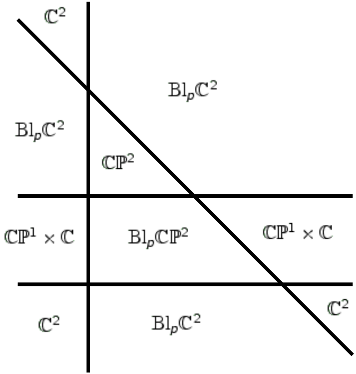
$(\mathbb{C}^*)^n$ acts naturally on $T^*\mathbb{C}^n$. Taking a hyperkähler quotient gives a hypertoric variety $\mathfrak{M} := T^*\mathbb{C}^n /\!/\!/\!/_\eta T_{V^\perp}$.

Some properties:

- complex symplectic orbifold
- $\dim_{\mathbb{C}} \mathfrak{M} = 2 \dim V$
- residual torus $T := (\mathbb{C}^*)^n / T_{V^\perp}$ acts symplectically
- ξ defines a cocharacter $\mathbb{C}^* \to T$, giving a 1-parameter flow on $\mathfrak{M}$.

The extended core

The extended core of $\mathfrak{M}$ is a Lagrangian subvariety whose components are the toric varieties X_α whose moment polyhedra are the chambers of the arrangement, glued along toric divisors just as the chambers are.

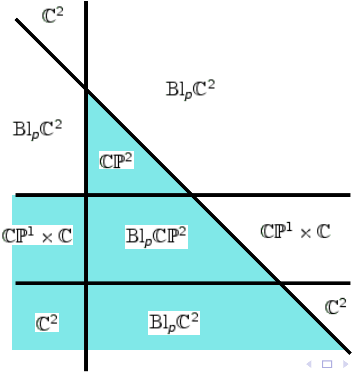


The ξ-bounded core

The stable set for the flow given by ξ:

$$\mathfrak{X} := \{x \in \mathfrak{M} \mid \lim_{t \to 0} t \cdot x \text{ exists}\}$$

is the union $\bigcup_{\alpha \in \mathcal{P}} X_\alpha$ of core components where $\mathcal{P}$ = the set of bounded feasible regions.



The convolution algebra

Let $\widetilde{\mathfrak{X}} =$ the disjoint union $\coprod_{\alpha\in\mathcal{P}} X_\alpha$, $\pi : \widetilde{\mathfrak{X}} \longrightarrow \mathfrak{M}$ the morphism induced by the inclusions $X_\alpha \hookrightarrow \mathfrak{M}$.
Look at the fiber product

$$\widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}} = \coprod_{(\alpha,\beta)\in\mathcal{P}\times\mathcal{P}} X_\alpha \cap X_\beta.$$

Its cohomology $B = B(V, \eta, \xi) := H^*(\widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}}, \mathbb{C})$ carries a noncommutative convolution product

$$a \star b = p_{13*}\big(p_{12}^*(a) \cup p_{13}^*(b)\big).$$

Where $p_{12}, p_{13}, p_{23} : \widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}} \longrightarrow \widetilde{\mathfrak{X}} \times_\pi \widetilde{\mathfrak{X}}$ denote the natural projections.

The convolution algebra: grading

The multiplication $\star$ is not a graded map for the usual grading.

Instead, if we let $d_{\alpha\beta} = \dim_{\mathbb{C}} \mathfrak{X} - \dim_{\mathbb{C}}(X_\alpha \cap X_\beta)$, then setting

$$deg(a) = d(a) + d_{\alpha\beta}$$

for

$$a \in H^{d(a)}(X_\alpha \cap X_\beta, \mathbb{C}).$$

makes B into a non-negatively graded algebra.

B is quadratic

With this grading, B is a quadratic algebra.

B_0 is spanned by commuting idempotents $e_\alpha = 1_{\alpha\alpha} \in H^0(X_\alpha \cap X_\alpha)$.

B is generated over B_0 by B_1, which is spanned by elements $1_{\alpha\beta} \in H^0(X_\alpha \cap X_\beta)$ where $d_{\alpha\beta} = 1$ (equivalently, α and β share a codimension 1 face).

The remaining relations are in degree 2, given by

$$\mathrm{Ker}(B_1 \otimes_{B_0} B_1 \longrightarrow B_2).$$

Main Theorem

We can also form a hypertoric variety $\mathfrak{M}^\vee$ using the Gale dual arrangement $(V^\perp, -\xi, -\eta)$. Let $B^\vee$ be the corresponding convolution algebra.

Theorem

1. *B is Koszul.*
2. *B and $B^\vee$ are Koszul dual rings: $B^\vee \cong Ext^\bullet_B(B_0, B_0)$*
3. *The center $Z(B)$ is isomorphic to $H^\bullet(\mathfrak{M})$.*

Koszulity means: $Ext^i_B(B_0, B_0[j]) = 0$ unless $i = j$. It implies that B is quadratic.

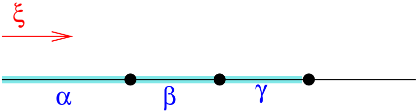
Quadratic duality

Given that they are Koszul rings, saying that B and $B^\vee$ are Koszul dual is equivalent to saying that they are quadratic dual rings:

- The idempotents e_α are in bijection.
- The degree 1 parts of $e_\alpha B e_\beta$ and $e_\alpha B^\vee e_\beta$ are dually paired.
- The degree 2 relations for B and $B^\vee$ are orthogonal complements of each other.

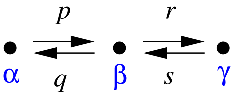
The first two properties are immediate from the fact that Gale duality is linear programming duality.

Example



$\mathfrak{M}$ is minimal resolution of singularities of $\mathbb{C}^2/\mathbb{Z}_3$
$\widetilde{\mathfrak{X}} = \mathbb{C} \sqcup \mathbb{CP}^1 \sqcup \mathbb{CP}^1$.

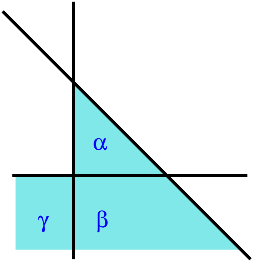
B is the path algebra of the quiver



modulo the relations

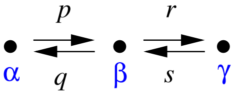
$$qp = 0, \quad pq = sr, \quad rp = 0, \quad qs = 0.$$

The dual example



$$\mathfrak{M}^\vee = T^*\mathbb{P}^2$$

$B^\vee$ is the path algebra of the same quiver



modulo the relations

$$pq + sr = 0, \quad rs = 0.$$

This result is analogous to the Koszul duality for category $\mathcal{O}$ from representation theory. In particular, objects of category $\mathcal{O}$ can be viewed as modules for a quantization of the structure sheaf of $T^*(G/B)$ which are supported on $\mathfrak{X}$, the conormal variety to the stratification of G/B by B-orbits. This is also the stable variety for a dominant cocharacter of the maximal torus.

(More precisely, it is $B^\vee$-modules which should be viewed as a "category $\mathcal{O}$" for $\mathfrak{M}$. Modules over $B^\vee$ can be viewed as modules over a quantization of $\mathfrak{M}$, supported on $\mathfrak{X}$).

Other features of category $\mathcal{O}$ carry over, too. In particular the category of B-modules is a highest weight category.

Symplectic duality

The relation between the hypertoric varieties $\mathfrak{M}$ and $\mathfrak{M}^\vee$ is an example of symplectic duality, a conjectural relation between pairs of symplectic varieties. Other varieties that should be symplectic dual are resolutions of slices to nilpotent orbit closures in type A. We do not have a complete definition of this duality, but some of the features symplectic dual varieties should satisfy are:

- $\mathfrak{M}$, $\mathfrak{M}^\vee$ have actions of algebraic tori T, $T^\vee$ with $\mathrm{Lie}(T) = H^2(\mathfrak{M}^\vee)$, $\mathrm{Lie}(T^\vee) = H^2(\mathfrak{M})$.
- There is a natural bijection between the fixed point sets $\mathfrak{M}^T$ and $(\mathfrak{M}^\vee)^{T^\vee}$

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