Math 704 – Homework #5

(1) Let S^n be the sphere in $\mathbb{R}^{n+1}$ with the usual metric.

(a) If (M, g) and $(\widetilde{M}, \tilde{g})$ are Riemannian manifolds, a map $\phi\colon M \to \widetilde{M}$ is a *local isometry* if each $p \in M$ has a neighborhood which is carried isometrically onto an open subset of $\widetilde{M}$. Suppose M is connected, and suppose that $\phi, \psi\colon M \to \widetilde{M}$ are local isometries such that for some $p \in M$, $\phi(p) = \psi(p)$ and $\phi_{*,p} = \psi_{*,p}$. Prove that $\phi = \psi$.

(b) Prove that if $F\colon S^n \to S^n$ is an isometry then F is the restriction of an orthogonal linear transformation of $\mathbb{R}^{n+1}$.

(2) Write the differential equation that a geodesic must satisfy for the plane $\mathbb{R}^2$ with the metric $g = dx^2 + (f(x))^2\, dy^2$, where $f \in \mathbb{C}^\infty(\mathbb{R})$ is a nowhere-vanishing smooth function. Find at least one solution to this equation.

(3) Let $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, with the Poincaré metric $g = dx^2 + y^{-2}\, dy^2$. Show that the Möbius transformation

$$z \mapsto \frac{az + b}{cz + d}$$

is an isometry of M, where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$, and $z = x + iy \in \mathbb{C}$.

(4) Let (M, g) be a Riemannian manifold, and ∇ the Riemannian connection. Given a smooth function $f\colon M \to \mathbb{R}$, define its Hessian H_f by

$$H_f(X, Y) = X(Yf) - (\nabla_X Y)f,$$

where $X, Y \in \mathcal{T}(M)$ are smooth vector fields on M.

(a) Show that H_f defines a symmetric $(0, 2)$-tensor field on M.

(b) If $df_p = 0$ for $p \in M$, then $H_f(p)\colon T_pM \times T_pM \to \mathbb{R}$ does not depend on the Riemannian metric.

(5) Let (M, g) be a Riemannian manifold, and let ∇ be the Riemannian connection. Let $\{E_i\}$ be a local frame for TM over $U \subset M$, and let $\{\phi^i\}$ be the dual coframe.

(a) Show that there is a uniquely determined matrix $\{\omega_i^j\}$ of 1-forms so that

$$\nabla_X E_i = \omega_i^j(X) E_j.$$

They are called the *connection* 1*-forms of* ∇ relative to the frame (they can be defined for connections other than one coming from a Riemannian connection). Express these 1-forms in terms of the Christoffel symbols Γ_{ij}^k of ∇ with respect to the same frame.

(b) Suppose that E_i is an orthonormal frame. Show that $\omega_i^j = -\omega_i^j$. In particular, for a 2-manifold, the Riemannian connection can be described locally by a single 1-form ω_1^2.

(c) Write two vector fields in terms of the frame: $X = \phi^i(X)E_i$, $Y = \phi^i(Y)E_i$. Plug these into the equation $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ and deduce *Cartan's first structure equation*:

$$d\phi^j = \phi^i \wedge \omega_i^j.$$

(remember the formula for $d\phi^j(X,Y)$!)

(d) To determine a 1-form it is enough to determine its wedge product with each element in a coframe. Thus the first structure equation can be used to compute the ω_i^j, and thus the connection. Apply this method to the following situation: $M = \mathbb{R}^2$, $g = \lambda^2 \bar{g}$, where $\bar{g}$ is the standard Euclidean metric, and $\lambda = \lambda(x,y) \in \mathbb{C}^\infty(M)$ is everywhere positive. You can use the orthonormal frame $\lambda^{-1}\partial/\partial x$, $\lambda^{-1}\partial/\partial y$.

(6) Use the same assumptions and definitions as the previous problem. Define a matrix of 2-forms $\Omega_i^j \in \mathcal{A}^2(M)$, called the *curvature 2-forms*, by

$$\Omega_i^j = \frac{1}{2} R_{kli}{}^j \phi^k \wedge \phi^l,$$

or in a more coordinate-free language, by the requirement that $R(X,Y)E_i = \Omega_i^j(X,Y)E_j$.

(a) Prove *Cartan's second structure equation:*

$$\Omega_i^j = d\omega_i^j + \omega_i^k \wedge \omega_k^j.$$

[Hint: expand out $R(E_k, E_l)E_i$.]

(b) Use this to continue the computation from part (d) of the previous problem: compute the 2-form Ω_2^1.

(c) Suppose $M \subset \mathbb{R}^3$ is an embedded surface in Euclidean space, with the induced metric. Let K be the Gaussian curvature of M. If the forms ω^i and Ω_i^j are computed relative to an orthonormal frame, show that

$$\Omega_1^2 = d\omega_1^2 = -K\phi^1 \wedge \phi^2.$$

(7) Let $t \mapsto (\phi(t), \psi(t))$ be an injective parametrized curve in the plane with unit speed, so $\dot{\phi}^2 + \dot{\psi}^2 \equiv 1$. Also suppose that $\psi(t) > 0$ for all t. Let $M \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating this curve around the x-axis. It can be described parametrically by

$$(s,t) \mapsto (\phi(t), \cos(s)\psi(t), \sin(s)\psi(t)).$$

For this surface, compute (1) the metric induced from the Euclidean metric on $\mathbb{R}^3$, (2) the Christoffel symbols with respect to the coordinate frame $\{\partial/\partial s, \partial/\partial t\}$, and (3) the Gaussian curvature of M.