Math 704 – Homework #4

(1) Lee, Chapter 14: problems 6, 7, 8, 9, 11 (on problem 6, a retraction of M onto ∂M is a map $r\colon M \to \partial M$ so that $r(x) = x$ for all $x \in \partial M$).

(2) Lee, Chapter 15: problems 3, 6.

(3) Let $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be given the Riemannian metric by

$$g(\partial_x, \partial_x) = g(\partial_y, \partial_y) = \lambda(x, y), \quad g(\partial_x, \partial_y) = 0,$$

with $\lambda(x, y) > 0$ everywhere on M (here $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$). Such a metric is called *conformally equivalent* to the Euclidean metric; it has the same angles between tangent vectors but not the same lengths.

Suppose that around every point in M there exist local coordinates (u^1, u^2) so that

$$g(\partial/\partial u^i, \partial/\partial u^j) = \delta_{ij}.$$

Show that this implies that $x + iy \mapsto u_1(x, y) + iu_2(x, y)$ is a complex analytic function (satisfies the Cauchy-Riemann equations). Use the theory of complex functions to conclude that $\log(\lambda)$ is a harmonic function.

(4) (a) Let α be a closed 1-form on S^4. Show that there exists a point $p \in S^4$ with $\alpha_p = 0$.
(b) Let β be a closed 2-form on S^4. Show that

$$\int_{S^4} \beta \wedge \beta = 0.$$

(5) (a) Let $M = \mathbb{R}^2 \setminus (\mathbb{Z} \times \{0\})$. Show that the deRham cohomology group $H^1(M)$ is infinite-dimensional.
(b) Suppose that a manifold M has a finite cover by open sets $U_1, \ldots, U_r$ so that the intersection of any number of them is either empty or else smoothly homotopy equivalent to a point (Such a cover always exists when M is compact). Show that $H^k(M)$ is finite-dimensional for all k.