Math 704 – Homework #3

(1) Lee, Chapter 12 problems 4, 6, and 8.

(2) Lee, Chapter 13 problems 1, 5, and 6.

(3) Let the surface $M \subset \mathbb{R}^3$ be the graph of a function $f \in C^\infty(\mathbb{R}^2)$. Compute an expression for the volume element Ω of M in the natural coordinates given by projecting back to $\mathbb{R}^2$.

(4) Suppose that V is a finite-dimensional vector space, and take $\alpha \in \Lambda^1(V) = V^*$ and $\beta \in \Lambda^k(V)$. Show that β is divisible by α (i.e. $\beta = \alpha \wedge \gamma$ for some $\gamma \in \Lambda^{k-1}(V)$) if and only if $\alpha \wedge \beta = 0$.

(5) Suppose that V is a vector space of dimension n, equipped with an orientation and a Riemannian structure (a positive definite bilinear form). Show that for each $0 \leq k \leq n$ there exists a unique linear operator $*\colon \Lambda^k(V) \to \Lambda^{n-k}(V)$ so that

$$*(v_1 \wedge \cdots \wedge v_k) = v_{k+1} \wedge \cdots \wedge v_n$$

whenever $v_1, \ldots, v_n$ is a positively oriented orthonormal basis of V^*. (Note that the Riemannian structure gives an identification of V with V^*, so V^* also inherits an orientation and bilinear structure).

Show that applying this operator twice gives $(-1)^{k(n-k)}$ times the identity.

If M is an oriented Riemannian manifold, then these operators $*_p$ for every $p \in M$ fit together to give an operator $*\colon \mathcal{A}^k(M) \to \mathcal{A}^{n-k}(M)$. Take for example $M = \mathbb{R}^n$ with the Euclidean metric and the standard orientation. Compute a formula for the map

$$*d*d\colon \mathbb{C}^\infty(\mathbb{R}^n) \to \mathbb{C}^\infty(\mathbb{R}^n).$$