Math 704 – Homework #2

(1) Lee, problems 11-3, 11-7, 11-9, 11-10, 11-13, 11-16, 11-19.

(2) For a finite-dimensional vector space V, construct a natural isomorphism $B(V,V) \to \mathrm{Hom}(V,V^*)$ (you can either use problem 11-3 or do it directly). What is the image of the subspace $\Sigma^2(V) \subset B(V,V) = V^* \otimes V^*$ of symmetric 2-tensors under this isomorphism?

(3) Even if the vector spaces in 11-3 are not finite dimensional, there is still a natural map $\Phi\colon V^* \otimes W \to \mathrm{Hom}(V,W)$. Show that is is not an isomorphism if V and W are not finite dimensional as follows. Let $V = W = \mathbb{R}^\infty = \bigcup_{n\geq 1} \mathbb{R}^n$ be the vector space of all sequences $(a_i)_{i\geq 1}$ which are eventually zero. Show that the identity map $V \to W$ is not in the image of Φ.

(4) Suppose M is a smooth manifold, an let $\alpha \in \mathcal{T}^1(M)$ be a covector field (covariant 1-tensor). Define a function $\beta\colon \mathcal{T}(M) \times \mathcal{T}(M) \to C^\infty(M)$ by the formula

$$\beta(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]).$$

Prove that this defines a covariant 2-tensor, i.e. a section of $T^2(M)$.

(5) If G is a Lie group, we say a Riemannian metric g on G (warning! Notation conflict!) is left-invariant if $L_h^* g = g$ for all $h \in G$, or in other words if $g((L_h)_*X, (L_h)_*Y) \circ L_h = g(X,Y)$ for all vector fields $X, Y \in \mathcal{T}(G)$ (as usual, $L_h\colon G \to G$ denotes left multiplication by h). Show that for every positive definite bilinear form g_e on T_eG there is a unique metric on G extending g_e.

Consider the group $G \subset GL_2(\mathbb{R})$ of invertible upper-triangular matrices, and use the global coordinate chart whose coordinate functions $x = a_{11}$, $y = a_{12}$, $z = a_{22}$ are the matrix entries. Compute explicitly the left-invariant metric whose value at the identity I is

$$(dx)_I^2 + (dy)_I^2 + (dz)_I^2.$$

(6) Let g be any Riemannian metric on $\mathbb{R}$. Show that $(\mathbb{R}, g)$ is isometric to $\mathbb{R}$ with the Euclidean metric dx^2.