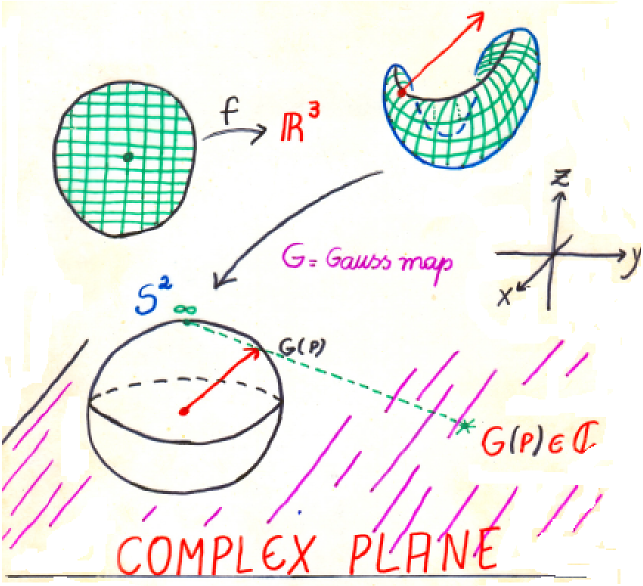
Definition of minimal surface

A surface $\mathbf{f}: \mathbf{M} \to \mathbf{R}^3$ is **minimal** if:

- $\mathbf{M}$ has **MEAN CURVATURE** $= \mathbf{0}$.
- Small pieces have **LEAST AREA.**
- Small pieces have **LEAST ENERGY.**
- Small pieces occur as **SOAP FILMS.**
- Coordinate functions are **HARMONIC.**
- Conformal Gauss map $\mathbf{G}: \mathbf{M} \to \mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$.
  MEROMORPHIC GAUSS MAP

Meromorphic Gauss map



Weierstrass Representation

Suppose $\mathbf{f}\colon \mathbf{M} \subset \mathbf{R}^3$ is minimal,

$$\mathbf{g}\colon \mathbf{M} \to \mathbf{C} \cup \{\infty\},$$

is the meromorphic Gauss map,

$$\mathbf{dh} = \mathbf{dx_3} + \mathbf{i} * \mathbf{dx_3},$$

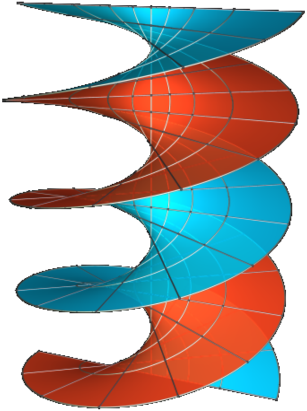
is the holomorphic height differential. Then

$$\mathbf{f(p)} = \mathbf{Re} \int^{\mathbf{p}} \frac{1}{2} \left[\frac{1}{\mathbf{g}} - \mathbf{g}, \frac{\mathbf{i}}{2} \left(\frac{1}{\mathbf{g}} + \mathbf{g} \right), 1 \right] \mathbf{dh}.$$

$$M = C$$

$$dh = dz = dx + i\,dy$$

$$g(z) = e^{iz}$$



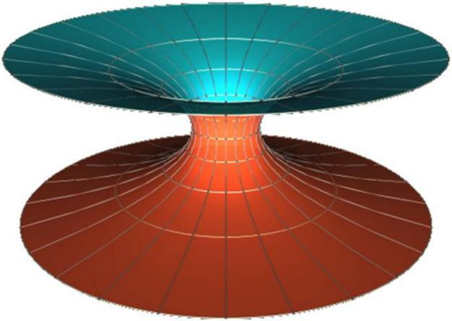
Helicoid

Image by Matthias Weber

$$\mathbf{M} = \mathbf{C} - \{(0,0)\}$$

$$\mathbf{dh} = \frac{1}{\mathbf{z}}\mathbf{dz}$$

$$\mathbf{g(z)} = \mathbf{z}$$



Finite topology minimal surfaces with 1 end

Theorem (Meeks, Rosenberg)

A complete, embedded, simply-connected minimal surface in $\mathbf{R}^3$ *is a plane or a helicoid.*

Theorem (Meeks, Rosenberg)

Every properly embedded, non-planar minimal surface in $\mathbf{R}^3$ *with finite genus and one end has the conformal structure of a compact Riemann surface* $\overline{\mathbf{M}}_g$ *of genus* g *minus one point, can be represented by meromorphic data on* $\overline{\mathbf{M}}_g$ *and is* **asymptotic** *to a* **helicoid**.

Finite topology minimal surfaces

Theorem (Collin)

If $\mathbf{M} \subset \mathbf{R}^3$ *is a properly embedded minimal surface with more than one end, then each annular end of* $\mathbf{M}$ *is* **asymptotic** *to the end of a* **plane** *or a* **catenoid***. In particular, if* $\mathbf{M}$ *has finite topology and more than one end, then* $\mathbf{M}$ *has finite total Gaussian curvature.*

Theorem (Meeks, Rosenberg)

Every properly embedded, non-planar minimal surface in $\mathbf{R}^3/\mathbf{G}$ *with finite genus has the conformal structure of a compact Riemann surface* $\overline{\mathbf{M}}_g$ *of genus* g *punctured in a finite number of points and can be represented by meromorphic data on* $\overline{\mathbf{M}}_g$*. Each annular end is* **asymptotic** *to the quotient of a half-helicoid (***helicoidal***), a plane (***planar***) or a half-plane (***Scherk type***).*

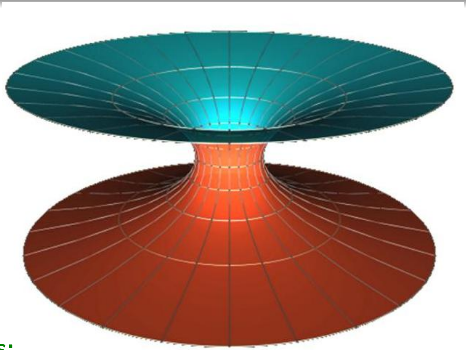
Properness of finite genus/topology examples

Theorem (Colding, Minicozzi)

A complete, embedded minimal surface of finite topology in $\mathbf{R}^3$ *is properly embedded.*

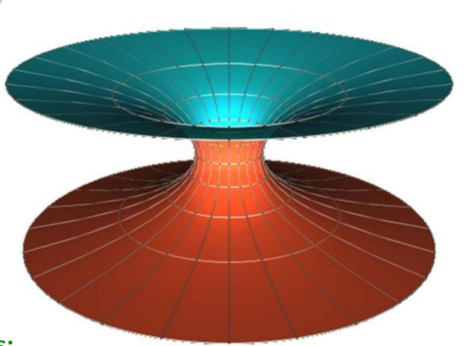
Theorem (Meeks, Perez, Ros)

A complete, embedded minimal surface of finite genus and a countable number of ends in $\mathbf{R}^3$ *or in* $\mathbf{R}^3/\mathbf{G}$ *is properly embedded.*



Key Properties:

- In 1741, **Euler** discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, then one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves.
- In 1776, **Meusnier** verified that the catenoid has zero mean curvature.
- This surface has genus zero, two ends and total curvature -4π.

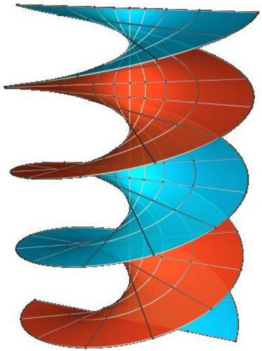


Key Properties:

- Together with the plane, the catenoid is the only minimal surface of revolution (**Euler** and **Bonnet**).
- It is the unique complete, embedded minimal surface with genus zero, finite topology and more than one end (**López** and **Ros**).
- The catenoid is characterized as being the unique complete, embedded minimal surface with finite topology and two ends (**Schoen**).

Helicoid.

Image by Matthias Weber

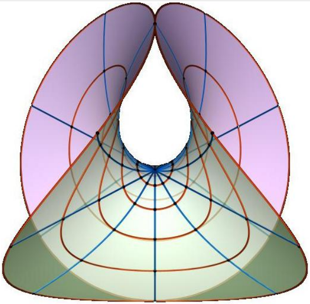


Key Properties:

- Proved to be minimal by **Meusnier** in 1776.
- The helicoid has genus zero, one end and infinite total curvature.
- Together with the plane, the helicoid is the only ruled minimal surface (**Catalan**).
- It is the unique simply-connected, complete, embedded minimal surface (**Meeks** and **Rosenberg**, **Colding** and **Minicozzi**).

Enneper surface.

Image by Matthias Weber

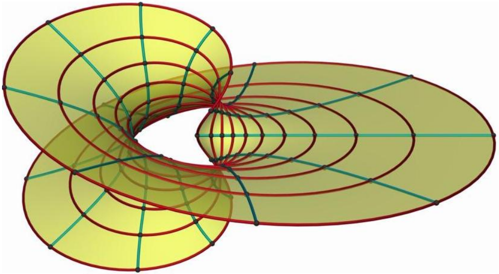


Key Properties:

- Weierstrass Data: $\mathbf{M} = \mathbf{C}$, $g(z) = z$, $dh = z\,dz$.
- Discovered by **Enneper** in 1864, using his newly formulated analytic representation of minimal surfaces in terms of holomorphic data, equivalent to the Weierstrass representation.
- This surface is non-embedded, has genus zero, one end and total curvature -4π.
- It contains two horizontal orthogonal lines and the surface has two vertical planes of reflective symmetry.

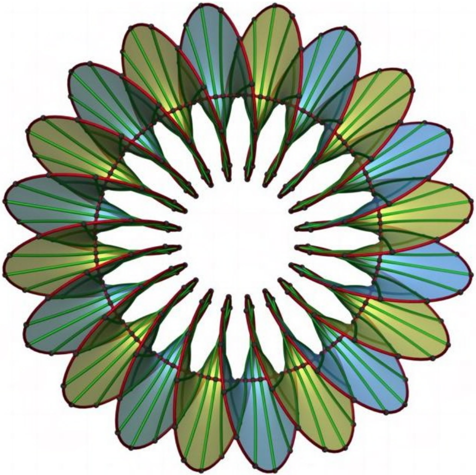
Meeks minimal Möbius strip.

Image by Matthias Weber



Key Properties:

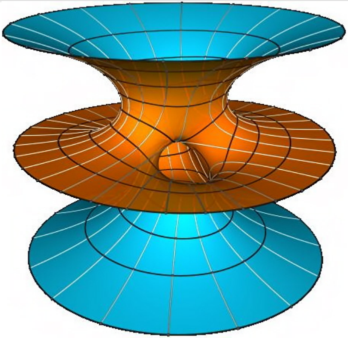
- Weierstrass Data: $\mathbf{M} = \mathbf{C} - \{0\}$, $g(z) = z^2 \left(\frac{z+1}{z-1}\right)$, $dh = i \left(\frac{z^2-1}{z^2}\right) dz$.
- Found by **Meeks** in 1981, the minimal surface defined by this Weierstrass pair double covers a complete, immersed minimal surface $\mathbf{M}_1 \subset \mathbf{R}^3$ which is topologically a Möbius strip.
- This is the unique complete, minimally immersed surface in $\mathbf{R}^3$ of finite total curvature -6π (**Meeks**).



Key Properties:

- Weierstrass Data: $\mathbf{M} = \mathbf{C} - \{0\}$, $\quad g(z) = -z\frac{z^n+i}{iz^n+i}$, $\quad dh = \frac{z^n+z^{-n}}{2z}dz$.
- Discovered in 2004 by **Meeks** and **Weber** and independently by **Mira**.

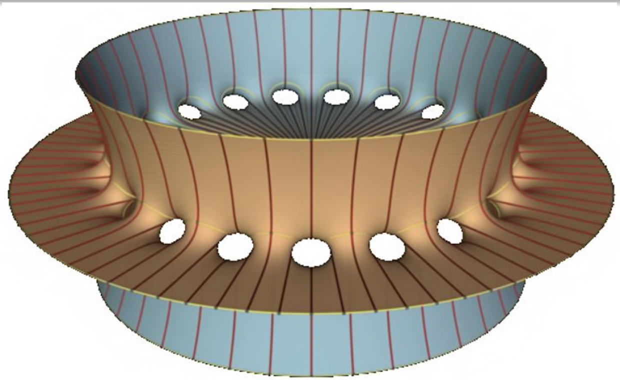
Costa torus. Image by Matthias Weber



Key Properties:

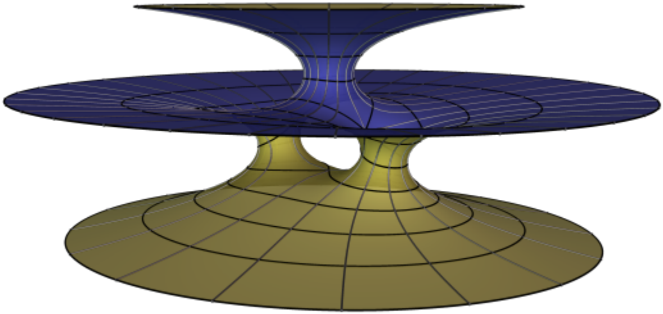
- Weierstrass Data: Based on the square torus
  $\mathbf{M} = \mathbf{C}/\mathbf{Z}^2 - \{(\mathbf{0},\mathbf{0}),(\frac{1}{2},\mathbf{0}),(\mathbf{0},\frac{1}{2})\}, \quad g(z) = \mathcal{P}(z)$.
- Discovered in 1982 by **Costa**.
- This is a thrice punctured torus with total curvature -12π, two catenoidal ends and one planar middle end. **Hoffman** and **Meeks** proved its global embeddedness.
- The Costa surface contains two horizontal straight lines l_1, l_2 that intersect orthogonally, and has vertical planes of symmetry bisecting the right angles made by l_1, l_2.

Costa-Hoffman-Meeks surfaces. Image by M. Weber



Key Properties:

- Weierstrass Data: Defined in terms of cyclic covers of $\mathbb{S}^2$.
- These examples $\mathbf{M}_k$ generalize the Costa torus, and are complete, embedded, genus k minimal surfaces with two catenoidal ends and one planar middle end. Both existence and embeddedness were given by **Hoffman** and **Meeks**.



Key Properties:

- The Costa surface is defined on a square torus $\mathbf{M}_{1,1}$, and admits a deformation (found by **Hoffman** and **Meeks**, unpublished) where the planar end becomes catenoidal.
- For any $a \in (0, \infty)$, take $\mathbf{M} = \mathbf{M}_{1,a}$ (which varies on arbitrary rectangular tori), $a = 1$ gives the Costa torus.
- **Hoffman** and **Karcher** proved existence/embeddedness.

Genus-one helicoid.

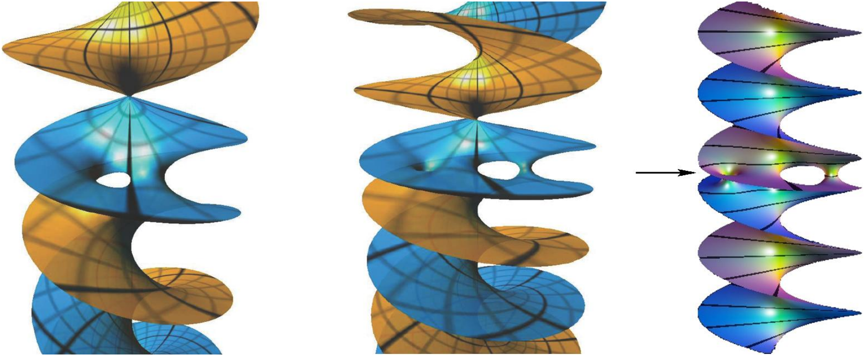
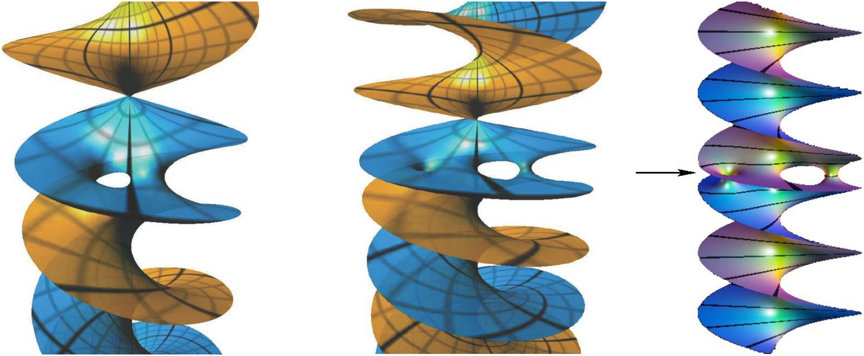


Figure: Left: The genus one helicoid. Center and Right: Two views of the (possibly existing) genus two helicoid. The arrow in the figure at the right points to the second handle. Images courtesy of M. Schmies (left, center) and M. Traizet (right).

Key Properties:

- **M** is conformally a certain rhombic torus T minus one point E. If we view T as a rhombus with edges identified in the usual manner, then E corresponds to the vertices of the rhombus.
- The diagonals of T are mapped into perpendicular straight lines contained in the surface, intersecting at a single point in space.

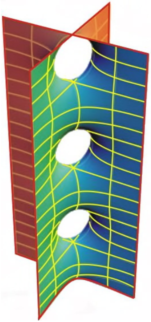
Genus-one helicoid.



Key Properties:

- The unique end of **M** is asymptotic to a helicoid, so that one of the two lines contained in the surface is an *axis* (like in the genuine helicoid).
- The Gauss map g is a meromorphic function on $T - \{E\}$ with an essential singularity at E, and both dg/g and dh extend meromorphically to T.
- Discovered in 1993 by **Hoffman**, **Karcher** and **Wei**.
- Proved embedded in 2007 by Hoffman, Weber and Wolf.

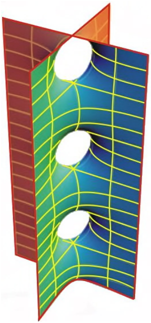
Singly-periodic Scherk surfaces. Image by M. Weber



Key Properties:

- Weierstrass Data: $\mathbf{M} = (\mathbf{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$, $dh = \frac{iz\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, for fixed $\theta \in (0, \pi/2]$.
- Discovered by **Scherk** in 1835, these surfaces denoted by $\mathcal{S}_\theta$ form a 1-parameter family of complete, embedded, genus zero minimal surfaces in a quotient of $\mathbf{R}^3$ by a translation, and have four annular ends.
- Viewed in $\mathbf{R}^3$, each surface $\mathcal{S}_\theta$ is invariant under reflection in the (x_1, x_3) and (x_2, x_3)-planes and in horizontal planes at integer heights, and can be thought of geometrically as a desingularization of two vertical planes forming an angle of θ.

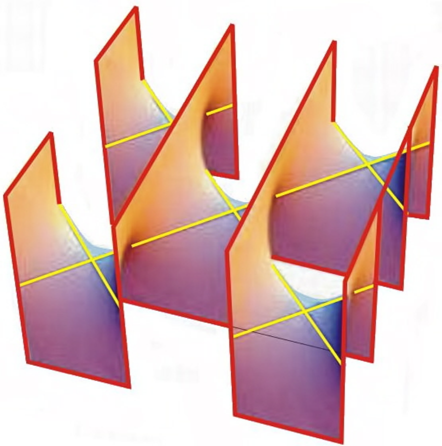
Singly-periodic Scherk surfaces. Image by M. Weber



Key Properties:

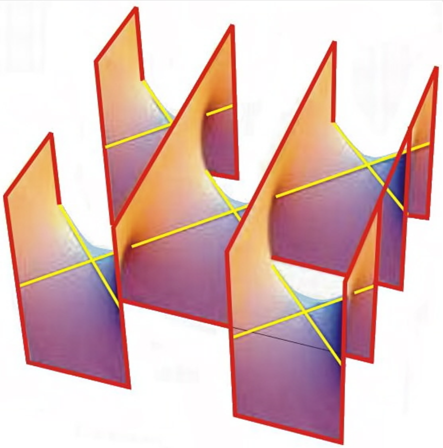
- The special case $\mathcal{S}_{\theta=\pi/2}$ also contains pairs of orthogonal lines at planes of half-integer heights, and has implicit equation $\sin z = \sinh x \sinh y$.
- Together with the plane and catenoid, the surfaces $\mathcal{S}_\theta$ are conjectured to be the only connected, complete, immersed, minimal surfaces in $\mathbf{R}^3$ whose area in balls of radius R is less than $2\pi R^2$. This conjecture was proved by **Meeks** and **Wolf** under the additional hypothesis of infinite symmetry.

.



Key Properties:

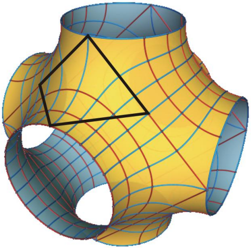
- Weierstrass Data: $\mathbf{M} = (\mathbf{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$, $dh = \frac{z\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, where $\theta \in (0, \pi/2]$ (the case $\theta = \frac{\pi}{2}$.
- It has implicit equation $e^z \cos y = \cos x$.
- Discovered by **Scherk** in 1835, are the conjugate surfaces to the singly-periodic Scherk surfaces.

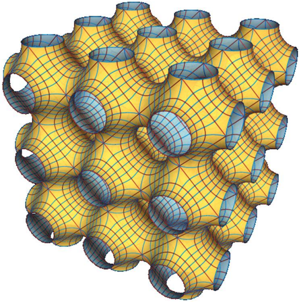


Key Properties:

- These surfaces are doubly-periodic with genus zero in their corresponding quotient $T^2 \times \mathbb{R}$ of $\mathbf{R}^3$, and were characterized by **Lazard-Holly** and **Meeks** as being the unique properly embedded minimal surfaces with genus zero in any $T^2 \times \mathbb{R}$.

Schwarz Primitive triply-periodic surface. Image by Weber

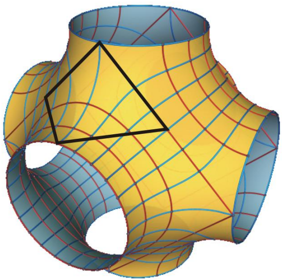


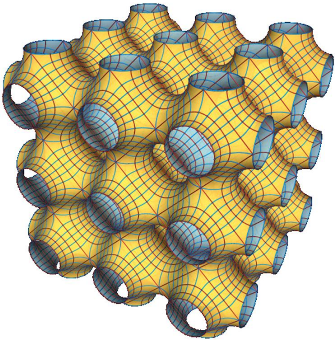


Key Properties:

- Weierstrass Data: $\mathbf{M} = \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2 \mid w^2 = z^8 - 14z^4 + 1\}$, $g(z, w) = z, \quad dh = \frac{z\,dz}{w}$.
- Discovered by **Schwarz** in the 1880's, it is also called the P-surface.
- This surface has a rank three symmetry group and is invariant by translations in $\mathbb{Z}^3$.
- Such a structure, common to any triply-periodic minimal surface (**TPMS**), is also known as a **crystallographic cell** or **space tiling**. Embedded **TPMS** divide $\mathbf{R}^3$ into two connected components (called **labyrinths** in crystallography), sharing $\mathbf{M}$ as boundary (or **interface**) and interweaving each other.

Schwarz Primitive triply-periodic surface. Image by Weber

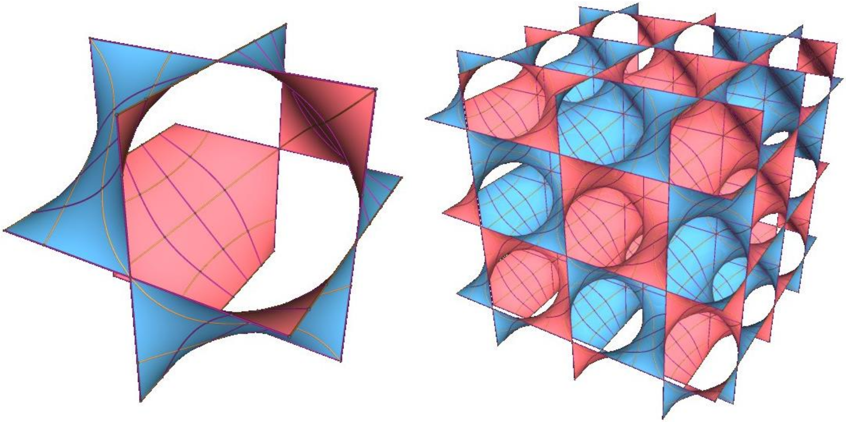




Key Properties:

- This property makes **TPMS** objects of interest to neighboring sciences as material sciences, crystallography, biology and others. For example, the interface between single calcite crystals and amorphous organic matter in the skeletal element in sea urchins is approximately described by the Schwarz Primitive surface.
- The piece of a **TPMS** that lies inside a crystallographic cell of the tiling is called a **fundamental domain.**

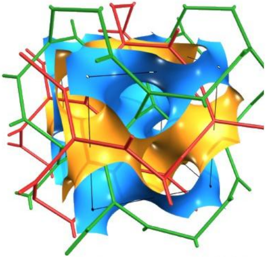
Schwarz Diamond surfaces.

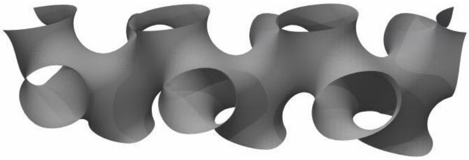


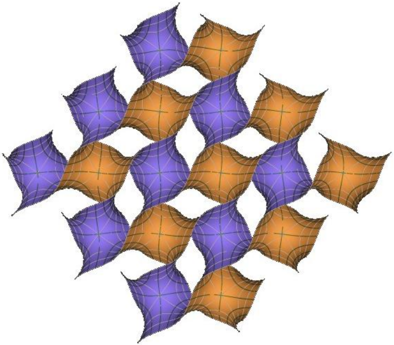
Discovered by Schwarz, it is the conjugate surface to the P-surface, and is another famous example of an embedded TPMS.

Schoen's triply-periodic Gyroid surface. Image by Weber

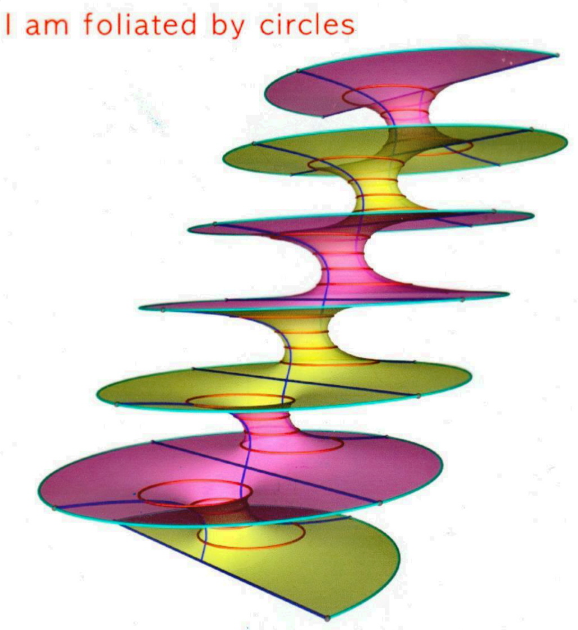
In the 1960's, **Schoen** made a surprising discovery: another minimal surface locally isometric to the Primitive and Diamond surface is an embedded TPMS, and named this surface the **Gyroid.**



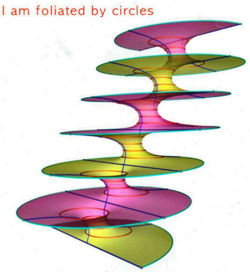




1860 Riemann's discovery! Image by Matthias Weber



Riemann minimal examples. Image by Matthias Weber



Key Properties:

- Discovered in 1860 by **Riemann**, these examples are invariant under reflection in the (x_1, x_3)-plane and by a translation T_λ, and in the quotient space $\mathbf{R}^3 / T_\lambda$ have genus one and two planar ends.
- After appropriate scalings, they converge to catenoids as $t \to 0$ or to helicoids as $t \to \infty$.
- The Riemann minimal examples have the amazing property that every horizontal plane intersects the surface in a circle or in a line.
- **Meeks**, **Pérez** and **Ros** proved these surfaces are the only properly embedded minimal surfaces in $\mathbf{R}^3$ of genus 0 and infinite topology.

KMR doubly-periodic tori.

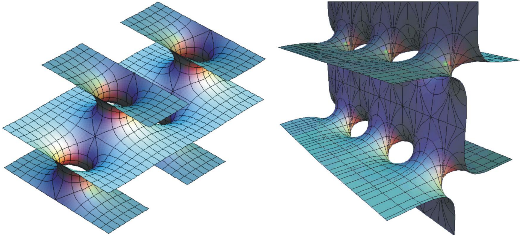
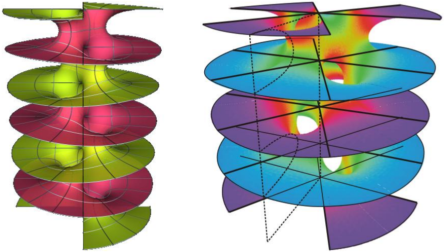


Figure: Two examples of doubly-periodic KMR surfaces. Images taken from the 3D-XplorMath Surface Gallery

Key Properties:

- The conjugate surface of any **KMR** surface also lies in this family.
- The first **KMR** surfaces were found by **Karcher** in 1988. At the same time **Meeks** and **Rosenberg** found examples of the same type as **Karcher's**.
- In 2005, **Pérez**, **Rodríguez** and **Traizet** gave a general construction that produces all possible complete, embedded minimal tori with parallel ends in any $T^2 \times \mathbb{R}$, and proved that this moduli space reduces to the three-dimensional family of **KMR** surfaces.



Key Properties:

- In 1989, **Callahan**, **Hoffman** and **Meeks** generalized the Riemann minimal examples by constructing for any integer $k \geq 1$ a singly-periodic, properly embedded minimal surface $\mathbf{M}_k \subset \mathbf{R}^3$ with infinite genus and an infinite number of horizontal planar ends at integer heights and are invariant under the orientation preserving translation by vector $T = (0, 0, 2)$, such that $\mathbf{M}_k/T$ has genus $2k + 1$ and two ends.
- Every horizontal plane at a non-integer height intersects $\mathbf{M}_k$ in a simple closed curve.
- Every horizontal plane at an integer height intersects $\mathbf{M}_k$ in $k + 1$ straight lines that meet at equal angles along the x_3-axis.

Introduction and history of the problem

Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.
$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

Introduction and history of the problem

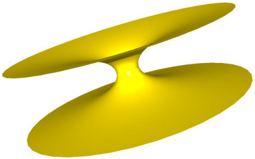
Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.
$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

Introduction and history of the problem

Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.
$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

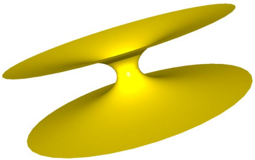


Introduction and history of the problem

Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.
$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

Collin, 1997: Finite topology and $\mathbf{k} > \mathbf{1} \Rightarrow$ finite total curvature.



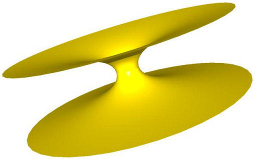
Introduction and history of the problem

Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.
$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

Collin, 1997: Finite topology and $\mathbf{k} > \mathbf{1}$ $\Rightarrow$ finite total curvature.

Colding-Minicozzi, 2004: limits of simply connected minimal surfaces = minimal laminations.



Introduction and history of the problem

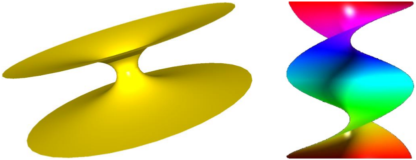
Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.
$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

Collin, 1997: Finite topology and $\mathbf{k} > \mathbf{1}$ $\Rightarrow$ finite total curvature.

Colding-Minicozzi, 2004: limits of simply connected minimal surfaces = minimal laminations.

Meeks-Rosenberg, 2005: $\mathbf{k} = \mathbf{1}$ $\Rightarrow$ plane, helicoid.



Introduction and history of the problem

Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.

$\mathbf{k} = \#\{\text{ends}\}$

López-Ros, 1991: Finite total curvature $\Rightarrow$ plane, catenoid

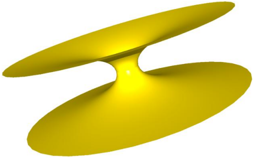
Collin, 1997: Finite topology and $\mathbf{k} > \mathbf{1}$ $\Rightarrow$ finite total curvature.

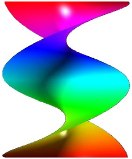
Colding-Minicozzi, 2004: limits of simply connected minimal surfaces = minimal laminations.

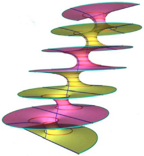
Meeks-Rosenberg, 2005: $\mathbf{k} = \mathbf{1}$ $\Rightarrow$ plane, helicoid.

Theorem (Meeks, Pérez, Ros, 2007)

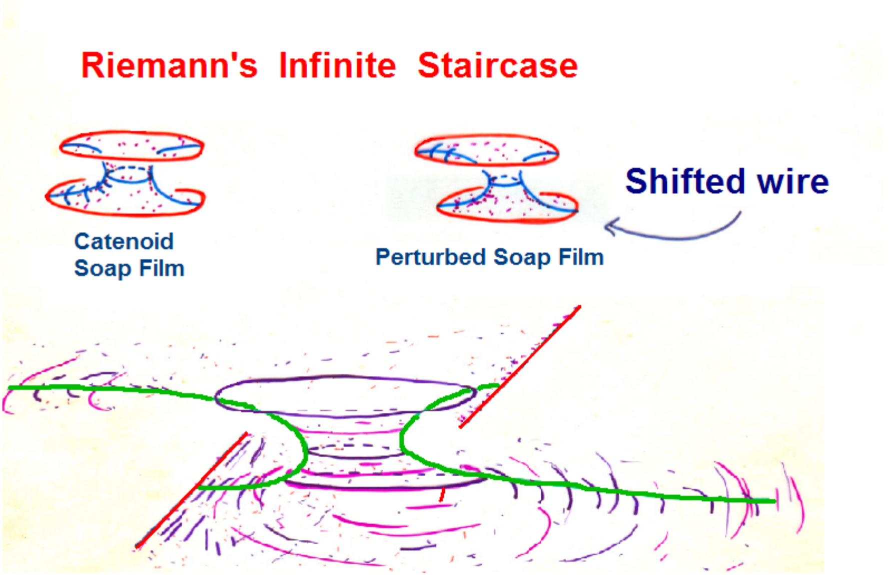
$\mathbf{k} = \infty$ $\Rightarrow$ **Riemann minimal examples.**



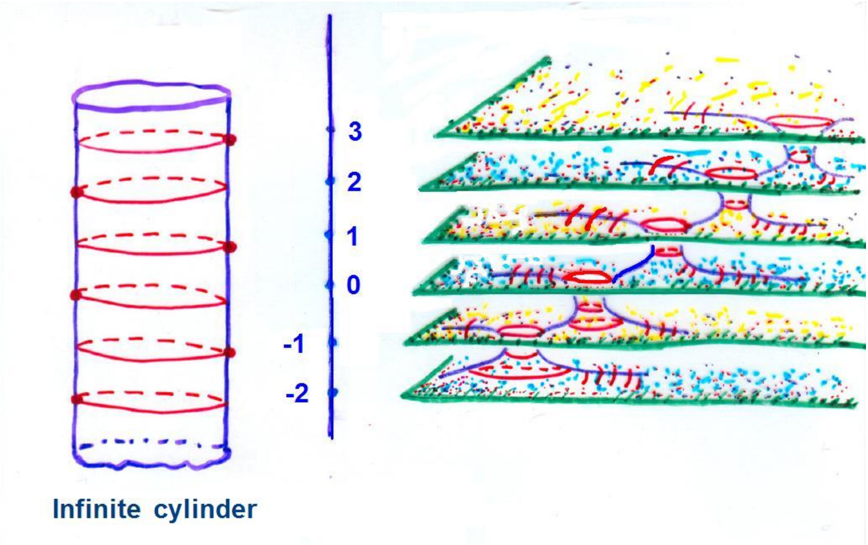


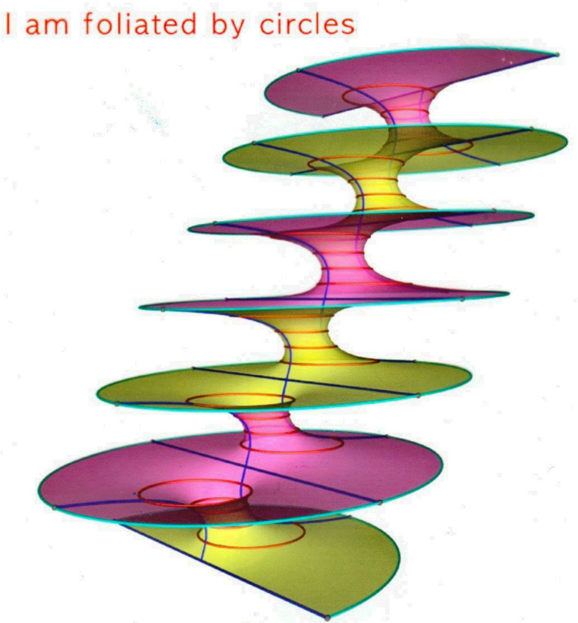


The family $\mathcal{R}_t$ of Riemann minimal examples

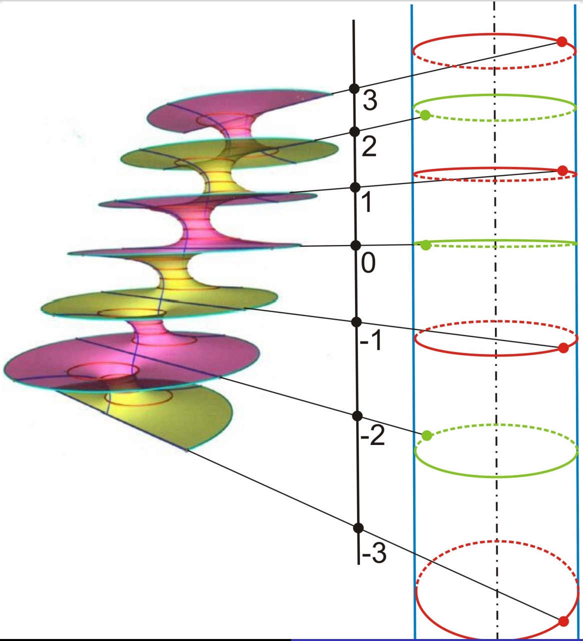


Cylindrical parametrization of a Riemann minimal example

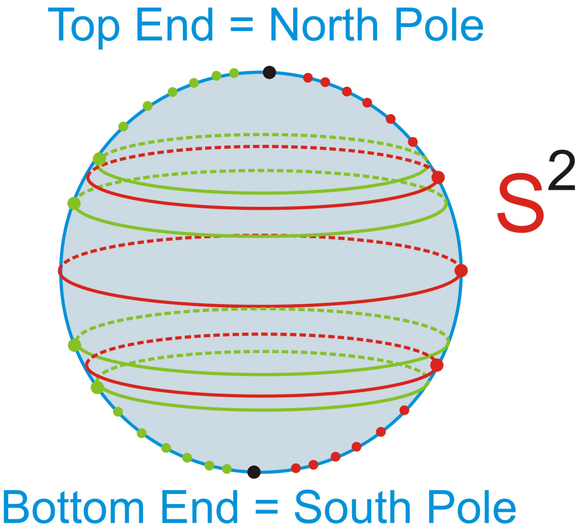



I am foliated by circles

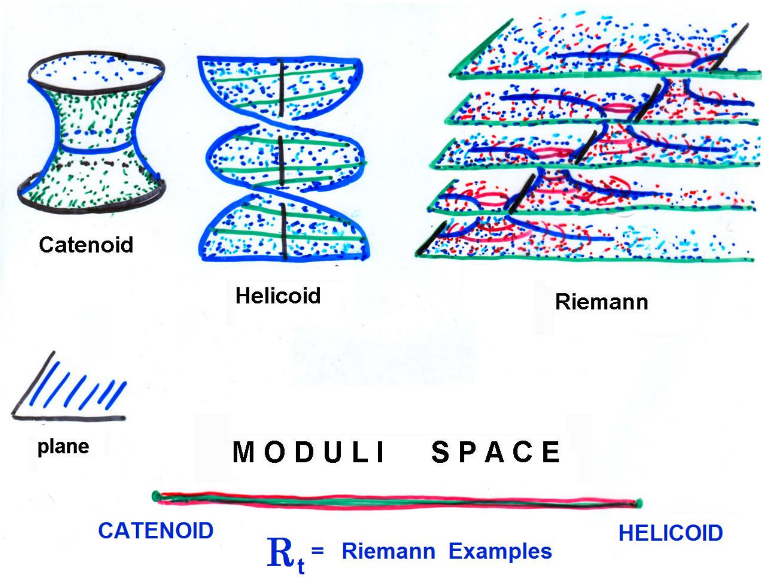
Cylindrical parametrization of a Riemann minimal example



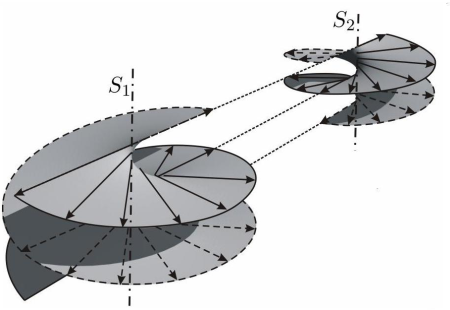
Conformal compactification of a Riemann minimal example

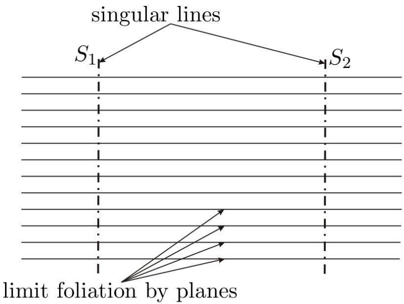


The moduli space of genus-zero examples



Riemann minimal examples near helicoid limits





Theorem (Meeks, Perez and Ros)

A **PEMS** *in* $\mathbf{R}^3$ *with genus zero and infinite topology is a Riemann minimal example.*

We now outline the main steps of the proof of this theorem.

Throughout this outline,

$\mathbf{M} \subset \mathbf{R}^3$ denotes a **PEMS** with genus zero and infinite topology.

Step 1: Control the topology of M

Theorem (Frohman-Meeks, C-K-M-R)

Let $\mathbf{\Delta} \subset \mathbf{R}^3$ *be a* **PEMS** *with an infinite set of ends* $\mathcal{E}$. *After a rotation of* $\mathbf{\Delta}$,

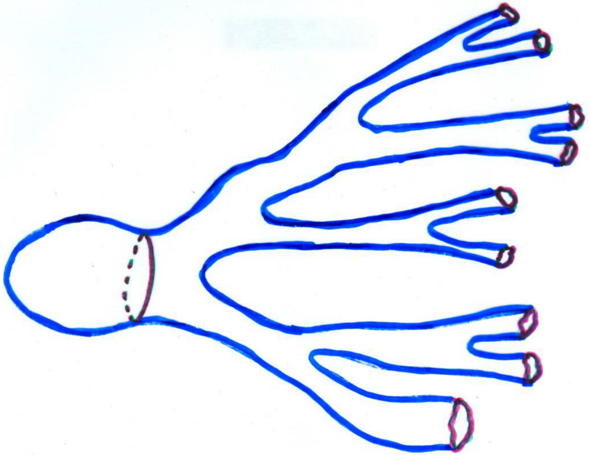
- $\mathcal{E}$ *has a natural linear ordering by relative heights of the ends over the xy-plane;*
- $\mathbf{\Delta}$ *has one or two limit ends, each of which must be a top or bottom end in the ordering.*

Theorem (Meeks, Perez, Ros)

The surface **M** *has* **two** *limit ends.*

Idea of the proof M has 2 limit ends. One studies the possible singular minimal lamination limits of homothetic shrinkings of **M** to obtain a **contradiction** if **M** has only one limit end.

A proper $g = 0$ surface with uncountable # of ends



$\mathbf{S}^2$ – Cantor set

Step 2: Understand the geometry of M

M can be parametrized **conformally** as
$\mathbf{f}\colon (\mathbf{S}^1 \times \mathbf{R}) - \mathcal{E} \to \mathbf{R}^3$ with $f_3(\theta, t) = t$ so that:

- The **middle ends** $\mathcal{E} = \{(\theta_n, t_n)\}_{n \in \mathbb{Z}}$ are **planar**.
- **M** has **bounded curvature**, **uniform local area estimates** and is **quasiperiodic**.
- For each t, consider the plane curve $\gamma_t(\theta) = \mathbf{f}(\theta, t)$ with speed $\lambda = \lambda_t(\theta) = |\gamma_t'(\theta)|$ and geodesic curvature $\kappa = \kappa_t(\theta)$. Then the **Shiffman function** $\boxed{\mathbf{S_M} = \lambda \frac{\partial \kappa}{\partial \theta}}$ extends to a bounded analytic function on $\mathbf{S}^1 \times \mathbf{R}$.
- $\mathbf{S_M}$ is a **Jacobi function** when considered to be defined on **M**. $\boxed{(\boldsymbol{\Delta} - 2\mathbf{K_M})\,\mathbf{S_M} = 0.}$

Step 3: Prove the Shiffman function S_M is integrable

$\mathbf{S_M}$ is **integrable** in the following sense. There exists a family $\mathbf{M_t}$ of examples with $\mathbf{M}_0 = \mathbf{M}$ such that **the normal variational vector field to each $\mathbf{M_t}$ corresponds to $\mathbf{S_{M_t}}$**.

The proof of integrability of $\mathbf{S_M}$ depends on:

- $(\boldsymbol{\Delta} - 2\mathbf{K_M})$ has finite dimensional bounded kernel;
- $\mathbf{S_M}$ viewed as an infinitesimal variation of Weierstrass data defined on $\mathbf{C}$, can be formulated by the **KdV** evolution equation.

KdV theory completes proof of integrability.

The Korteweg-de Vries equation (KdV)

$$\dot{g}_s = \frac{i}{2}\left(g''' - 3\frac{g'g''}{g} + \frac{3}{2}\frac{(g')^3}{g^2}\right) \in T_g\mathcal{W} \text{ (Shiffman)}$$

Question: Can we integrate $\dot{g}_s$? (This solves the problem)

(mKdV) Miura transf (KdV)

$$\dot{g}_s \xrightarrow{x=g'/g} \dot{x} = \frac{i}{2}(x''' - \frac{3}{2}x^2x') \xrightarrow{u=ax'+bx^2} \dot{u} = -u''' - 6uu'$$

$$u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g}$$

KdV hierarchy (infinitesimal deformations of u)

$$\left.\begin{array}{l} \frac{\partial u}{\partial t_0} = -u' \\ \frac{\partial u}{\partial t_1} = -u''' - 6uu' \\ \frac{\partial u}{\partial t_2} = -u^{(5)} - 10uu''' - 20u'u'' - 30u^2u' \\ \vdots \end{array}\right\}$$

All flows commute: $\frac{\partial}{\partial t_n}\frac{\partial u}{\partial t_m} = \frac{\partial}{\partial t_m}\frac{\partial u}{\partial t_n}$

u algebro-geometric $\overset{\text{def}}{\Leftrightarrow} \exists n,\ \frac{\partial u}{\partial t_n} \in \mathrm{Span}\{\frac{\partial u}{\partial t_0}, \dots, \frac{\partial u}{\partial t_{n-1}}\}$

Step 4: Show $\mathbf{S_M} = 0$

The property that $\mathbf{S_M} = 0$ is equivalent to the property that **M** is foliated by circles and lines in horizontal planes.

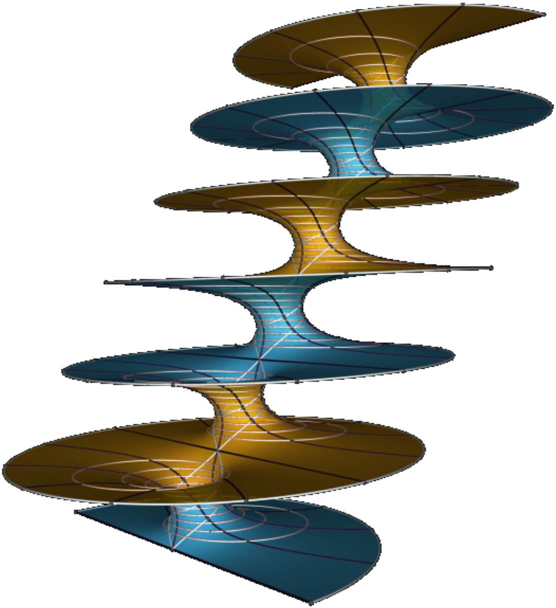
Theorem (Riemann 1860)

If **M** *is foliated by circles and lines in horizontal planes, then* **M** *is a Riemann minimal example.*

Holomorphic integrability of $\mathbf{S_M}$, together with the **compactness** of the moduli space of embedded examples, forces $\mathbf{S_M}$ to be **linear**, which requires the analytic data defining **M** to be **periodic**. In 1997, we proved that $\mathbf{S_M} = 0$ for periodic examples. Hence, **M is a Riemann minimal example.**

A Riemann minimal example

Image by Matthias Weber



1860 Riemann's discovery! Image by Matthias Weber