From Solitons to Dilatons: Relating Sine-Gordon Equations to Two-Dimensional Gravity

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Abstract

A soliton metric is a metric of the form $ds^2 = \cos^2\left(\frac{u}{2}\right) dx^2 \pm \sin^2\left(\frac{u}{2}\right) dt^2$, where $u = u(x,t)$ is a solid wave solution of a Sine-Gordon equation: $u_{xx} \pm u_{tt} = \pm A \sin(u)$. We explore the relationship between soliton metrics and black hole metrics in various two-dimensional models for classical gravity: Jackiw-Teitelboim Gravity (JT), String-Inspired Gravity (SIG), and Spherically Symmetric Gravity (SSG). In particular, we explore some concrete examples for which maps between these two spaces can be explicitly found and study a scalar field known as a dilaton, governing the black hole metric.

The talk has some physical connections, but it is primarily mathematical and is intended to be accessible to graduate students.

1 Introduction

A soliton metric is a metric of the form $ds^2 = cos^2(\frac{u}{2})dx^2 \pm \sin^2(\frac{u}{2})dt^2$, where $u(x,t)$ is a solid wave solution of a sine-Gordon equation: $u_{xx} \pm u_{tt} = \pm A\sin(u)$. We explore the relationship between soliton metrics and black hole metrics in various models for classical gravity in two spacetime dimensions.

One natural question is whether it is possible to find a general method for constructing maps between soliton metrics and black hole metrics; often, this involves solving a complicated system of highly nonlinear differential equations. As a simplification, we use the two-dimensional Jackiw-Teitelboim model for gravity to explore some concrete examples for which such maps can be explicitly found.

In addition, we provide a generalisation of the J-T action integral to extend some known results in the J-T case to the String-Inspired Theory (SIG) and the Spherically Symmetric Gravity (SSG) model.

Time permitting, we also study the construction of a pair of harmonic maps from $\mathbb{R}^2$ (equipped with a soliton metric) to $\mathbb{S}^2$. This leads us to a concrete connection between sigma-models, solitons and 2-dimensional gravity.

2 Background

2.1 Some notation

Let (M, g) be a pseudo-Riemannian manifold with $\dim(M) = m$ and suppose g is given locally by $ds^2 = \sum_{i,j=1}^m g_{ij} dx_i dx_j$. Then we shall use the following conventions to define the curvature tensor, Ricci tensor, and scalar curvature, noting that $g^{-1} = [g^{ij}]$:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^m g^{lk}\left[\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right] = \text{Christoffel Symbols} \tag{1}$$

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x_i} + \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_{p=1}^m [\Gamma_{ip}^l \Gamma_{jk}^p - \Gamma_{jp}^l \Gamma_{ik}^p] = \text{Curvature Tensor} \tag{2}$$

$$R_{ij} = \sum_{l=1}^m R_{ilj}^l = \text{Ricci Tensor} \tag{3}$$

$$-2K = R = \sum_{i,j=1}^m g^{ij} R_{ij} = \text{Scalar Curvature} \tag{4}$$

2.2 The two dimensional Einstein equation

Observe that when $m = 2$, $R_{ij} = (R/2)g_{ij}$, so that the Einstein vacuum equations

$$R_{ij} - \frac{R}{2} g_{ij} + \Lambda g_{ij} = 0 \tag{5}$$

are trivially satisfied for $\Lambda = 0$. Thus, one is motivated to examine the nontrivial Einstein equation $R = R(g) = A$, a constant, in two dimensions. This equation may be derived from the Jackiw-Teitelboim action

$$I_{JT}[\tau, g] = \frac{1}{2G}\int_M \sqrt{|det(g)|} dx_1 dx_2 (2A - R(g))\tau(x_1, x_2), \tag{6}$$

where the scalar field τ is varied. In particular, the field equations (or Euler-Lagrange equations) resulting from this action functional, upon varying the dilaton τ and the metric g are

$$\begin{aligned} R - 2A &= 0 \\ (\nabla_i \nabla_j - A g_{ij})\tau &= 0, \end{aligned} \tag{7}$$

where we recall $\nabla_i \nabla_j h = -\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial h}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\partial h}{\partial x_i}\right) = -\sum_{k=1}^m \Gamma_{ij}^k \frac{\partial h}{\partial x_k} + \frac{\partial^2 h}{\partial x_i \partial x_j}$.

Originally introduced as a Lagrange multiplier by Jackiw, variation of the dilaton demonstrates that solutions in JT gravity are metrics of constant curvature. Since the sine-Gordon equation has been used successfully to study surfaces of constant curvature, one is motivated to associate the two ideas. It turns out that the black hole metric coefficients can be expressed in terms of τ alone. This fact and the curvature condition allow us to closely relate black holes in JT gravity to soliton solutions of the Euclidean sine-Gordon equation.

2.3 Solitons and sine-Gordon equations

Observed in 1834 as a wavefront in a channel of water, engineer John Scott Russell remarked that he followed a "solitary wave" for over 2 miles before losing sight of it in the bends of the waterway; such waves were the subject of study in fluid mechanics, solid state physics, chemical and geological systems for the next 150 years.

A soliton is a localised large amplitude wave solutions of a nonlinear partial differential equations. It has particle-like, nondispersive properties: solitons propagate without spreading and can pass through each other, preserving their speed and shape after collision. One physical realisation of such waves is demonstrated by the motion of coupled pendulums on a mechanical transmission line, such as a spring. In this case, the motion is modeled by the classical sine-Gordon equation

$$u_{xx} - u_{tt} = m^2 \sin(u), \tag{8}$$

where $u = u(x,t)$ and $m > 0$ is a constant. This equation has soliton solutions describing collions between ideal solitary waves. As such, wave solutions to the sine-Gordon equation have been explored extensively in physics [1].

Note that in small amplitudes, the estimate $\sin(u) \approx (u)$ yields the Klein-Gordon equation.

2.4 Some explicit soliton solutions

It is known that the two-dimensional sine-Gordon equation is integrable, that is, it has infinitely many conservation laws. Hence, finding solutions to this wave equation using basic methods produces a variety of solitons. By assuming u is of the form $u = 4\arctan\left(\frac{F(x)}{G(t)}\right)$, we reduce equation 8 to a system of nonlinear ordinary differential equations in $F(x)$ and $G(t)$. Variations of the integration constants produce solutions of many forms. Some examples include (a, v, b, c constants):

kink, antikink (soliton, antisoliton): $u(x,t) = 4\arctan\left(e^{\pm\frac{m}{a}(x-vt)}\right)$, where $a^2 = 1 - v^2$.

[1] For instance, they have been studied as a possible model for elementary particles (Perring, Skyrme 1962)

breather: $u(x,t) = 4\arctan\left(\frac{a\sin(vmt)}{v\cosh(amx)}\right)$, where $a^2 = 1 - v^2$

kink-kink collision: $u(x,t) = 4\arctan\left(\frac{c\sinh(bmx)}{b\cosh(cmt)}\right)$, where $c^2 = b^2 - 1$

kink-antikink collision: $u(x,t) = 4\arctan\left(\frac{b\sinh(cmt)}{c\cosh(bmx)}\right)$, where $c^2 = b^2 - 1$

2.5 Relation to the scalar curvature equation

Let $u = u(x,t)$ be any (sufficiently smooth) function. Observe that if we construct the metric given by $ds^2 = \cos^2\left(\frac{u}{2}\right)dx^2 + \sin^2\left(\frac{u}{2}\right)dt^2$, then by direct computation its scalar curvature may be expressed in terms of u as $R = \frac{2(u_{xx}-u_{tt})}{\sin(u)}$. Thus, the metric solves the nontrivial Einstein equation $R = 2A$ if and only if u solves the sine-Gordon equation $u_{xx} - u_{tt} = A\sin(u)$.

It is also fruitful to consider the metric g given by

$$ds^2 = \cos^2\left(\frac{u}{2}\right)dx^2 - \sin^2\left(\frac{u}{2}\right)dt^2. \tag{9}$$

Again, we may compute the scalar curvature in terms of u to be

$$R(g) = \frac{2(u_{xx} + u_{tt})}{\sin(u)} = \frac{2\triangle u}{\sin(u)}, \tag{10}$$

so that $R(g) = 2A$ if and only if u solves the Euclidean sine-Gordon equation $\triangle u = A\sin(u)$.

In either case, when u is a solid wave solution of a sine-Gordon equation, we call the constructed metric above a *soliton metric*.

2.6 Black holes and the Schwarzschild metric

Solutions to the Einstein equations involve finding an appropriate metric on spacetime. Sometimes the metric found is in the form of a black hole. Heuristically, a black hole (or black hole metric) is a solution to the Einstein equations having a "horizon[2]." The first model of a black hole was given by Schwarzschild in 1916; it is an uncharged black hole with zero angular momentum, given by the Schwarzschild metric (M^4, g):

$$\begin{aligned} M^4 &= \mathbb{S}^3 \times \left\{ r \in \mathbb{R}^+ \,|\, 0 < r < 2M \right\} \\ ds^2 &= \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 - c^2\left(1 - \frac{2Gm}{c^2 r}\right) dT^2 + r^2 d\theta^2 + r^2\sin^2\varphi d\varphi^2. \end{aligned} \tag{11}$$

[2] In some region of spacetime, a distant observer can no longer obtain any information

This is the model from which we formulate our working definition of a black hole metric. In order to discuss the relationship between black hole metrics and soliton metrics, we restrict our attention to black holes lying in a plane, i.e. two spacetime dimensions. For our purposes then, we consider a black hole metric to be one of the form

$$ds^2 = f(r)dT^2 + f(r)^{-1}dr^2\,, \tag{12}$$

where $f(r)$ is a rational function of *solely* r. Our primary example, a two-dimensional slice of a three dimensional BTZ black hole metric, is modeled after the Schwarzschild metric above:

$$ds^2 \;=\; -\left(\Lambda r^2 - M + \frac{J^2}{4r^2}\right)dT^2 + \left(\Lambda r^2 - M + \frac{J^2}{4r^2}\right)^{-1} dr^2, \tag{13}$$

where J is an angular momentum term, and Λ, M are constants.

Upon computing the scalar curvature of this metric to be $R = 2\Lambda + \frac{3J^2}{2r^4}$, we immediately notice that R is constant if and only if the black hole has no angular momentum. In the case where $J = 0$ and $\Lambda = m^2$ we see that $R = 2m^2$, which is precisely the curvature of the soliton metric in 9: $ds^2 = \cos^2(\frac{u}{2})dx^2 - \sin^2(\frac{u}{2})dt^2$, where u is a solution of the Euclidean sine-Gordon equation $\triangle u = m^2\sin(u)$. Thus, the curvature condition in the JT field equations 7 is immediately satisfied by both of these metrics. This suggests that we may be able to use either metric to find an appropriate dilaton to solve the remaining equations of motion. Moreover, if it is possible to find a transformation of coordinates from the soltion metric to the black hole (or vice versa).

It is known that black holes are governed by mass, electric charge, and angular momentum alone (the "No Hair" theorems); in some sense, they are one of the simplest physical objects to study from the outside. However, much is unknown about their quantum and thermodynamic properties[3]. It is our motivation to provide a context by which one may use sine-Gordon solitons to restate and possibly address some of these problems [4].

3 Equations of Motion for τ

We shall denote (x,t) as the soliton coordinates and (T,r) as the black hole coordinates. There are two avenues by which we explore the dilaton equations in 7: $(\nabla_i\nabla_j - m^2 g_{ij})\tau = 0$.

[3] why the Bekenstein-Hawking entropy is so large, for instance

[4] for instance, the coordinate singularities in the sine-Gordon metric are of a different nature than those of the black hole metric

First, in soliton coordinates, we fix a metric g and find the dilaton τ. Next, in black hole coordinates, we choose a dilaton τ and find a metric g. Of course, transformations between the two must also be exploited, when possible.

3.1 The dilaton equations

Let $A = m^2$ and denote (x_1, x_2) by (x,t) so that we may restrict our attention to the specialised soliton metric g in 9 given in terms of $u = u(x,t)$ by $ds^2 = \cos^2(\frac{u}{2})dx^2 - \sin^2(\frac{u}{2})dt^2$, where u satisfies $\triangle u = m^2 \sin(u)$. Then by direct computation, we have (again) that $R = 2m^2 = 2A$. Thus, the equations of motion in 7 for the dilaton $\tau(x,t)$ reduce to

$$\begin{aligned}
\tau_{xx} + \frac{1}{2}\tan\left(\frac{u}{2}\right)u_x\tau_x + \frac{1}{2}\cot\left(\frac{u}{2}\right)u_t\tau_t - m^2\cos^2\left(\frac{u}{2}\right)\tau &= 0 \\
\tau_{tt} - \frac{1}{2}\tan\left(\frac{u}{2}\right)u_x\tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right)u_t\tau_t + m^2\sin^2\left(\frac{u}{2}\right)\tau &= 0 \\
\tau_{xt} + \frac{1}{2}\tan\left(\frac{u}{2}\right)u_t\tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right)u_x\tau_t &= 0 \qquad (14)
\end{aligned}$$

Thus, for a given choice of soliton $u(x,t)$, we may attempt to find a dilaton $\tau(x,t)$ and thus produce the solution (g, τ).

EXAMPLE 1. Let us first consider the simplest case, the kink soliton: $u(x,t) = 4\arctan\left(e^{\rho(x,t)}\right)$, where $\rho(x,t) = \frac{m}{a}(x - vt)$, $a^2 = 1 + v^2$. Observe that for any function ρ, we have the following trigonometric reductions $\cos(u/2) = -\tanh(\rho)$, $\sin(u/2) = \text{sech}(\rho)$. So the soliton metric can be rewritten in terms of hyperbolic functions. Then we have the following result.

THEOREM 1. *Let $u(x,t)$ be the kink soliton given above. For the soliton metric g given by*

$$\begin{aligned}
ds^2 &= \cos^2\left(\frac{u}{2}\right)dx^2 - \sin^2\left(\frac{u}{2}\right)dt^2 \\
&= \tanh^2(\rho)dx^2 - sech^2(\rho)dt^2, \qquad (15)
\end{aligned}$$

the dilaton $\tau(x,t) = asech(\rho(x,t))$ solves the field equations 14 (Gegenberg, Kunstatter). Moreover, there exists a coordinate transformation Ψ, having inverse Θ, taking 15 to the black hole metric

$$ds^2 = (v^2 - m^2r^2)dT^2 - (v^2 - m^2r^2)^{-1}dr^2 \quad (Williams). \qquad (16)$$

Proof. Using the reformulation of the metric g, we verify directly that the dilaton $\tau(x,t) = a\text{sech}(\rho(x,t))$ solves the field equations. The last statement is shown by directly verifying the coordinate transformation for the maps given by Williams (see references).

Let $\Psi(T,r) = (\psi_1(T,r), \psi_2(T,r))$ and $\Theta(x,t) = (\theta_1(x,t), \theta_2(x,t))$. Then

$$\begin{aligned} x = \psi_1(T,r) &= vT + \frac{1}{2m}\log\left[\frac{\sqrt{a^2 - m^2r^2}+1}{\sqrt{a^2-m^2r^2}-1}\right] \\ t = \psi_2(T,r) &= \frac{\psi_1}{v} - \frac{a}{mv} l \log\left[\frac{a+\sqrt{a^2-m^2r^2}}{mr}\right] \end{aligned} \tag{17}$$

provides the necessary change of coordinates. It is useful to note that the inverse of the above map $\Psi^{-1} := \Theta = (\theta_1, \theta_2)$ isolates the dilaton: τ. In fact,

$$\begin{aligned} T = \theta_1(x,t) &= -\frac{1}{2mv}\log\left[\frac{a\tanh(\rho(x,t))+1}{a\tanh(\rho(x,t))-1}\right] + \frac{x}{v} \\ r = \theta_2(x,t) &= \frac{a}{m}\operatorname{sech}(\rho(x,t)) \\ &= \frac{\tau(x,t)}{m} \end{aligned} \tag{18}$$

Note also that we are taking the soliton metric in 9 (or equivalently 15) to the black hole metric 16 with $J = 0$, $\Lambda = m^2$, and $M = v^2$. □

3.2 The breather solution

For the breather solution of the Euclidean sine-Gordon equation 8, an analogous result to Theorem 1 can be shown.

PROPOSITION 2. *Let* $u = 4\arctan(\frac{v\sinh(amx)}{a\cos(vmt)})$, $a^2 = 1+v^2$ *be a solution of the Euclidean sine-Gordon equation 8. Then for the soliton metric* g *given by* $ds^2 = \cos^2(\frac{u}{2})dx^2 - \sin^2(\frac{u}{2})dt^2$, *the dilaton*

$$\tau = \frac{4v^2am\sin(vmt)\sinh(amx)}{a^2\cos^2(vmt) + v^2\sinh^2(amx)} \tag{19}$$

solves the field equations.

Proof. We may construct τ by making an ansatz. Observe that if we sum the first two equations in 14, we have that τ must satisfy a linearisation of the Sine-Gordon equation, namely $\triangle\tau = m^2\cos(u)\tau$. Given that $\triangle u = m^2\sin(u)$, we make the ansatz $\tau = Au_t + Bu_x$, as

$$\begin{aligned}\triangle\tau &= \tau_{xx}+\tau_{tt}\\ &= Au_{txx}+Bu_{xxx}+Au_{ttt}+Bu_{xtt}\\ &= A(\triangle u)_t+B(\triangle u)_x\\ &= A(m^2\sin(u))_t+B(m^2\sin(u))_x\\ &= Am^2\cos(u)u_t+Bm^2\cos(u)u_x\\ &= m^2\cos(u)\tau. \end{aligned} \tag{20}$$

A reduction of the field equations using this ansatz allows us to solve[5] for τ. We thus obtain $\tau = \frac{4v^2 amA\sin(vmt)\sinh(amx)}{a^2\cos^2(vmt)+v^2\sinh^2(amx)}$. Thus, the pair (g,τ) solve the second J-T formulation of the Einstein equations. □

Unfortunately, in order to determine a transformation to a black hole metric, one must work a little harder. One known result (Gegenberg, Kunstatter) is that the target metric can be expressed in terms of the dilaton by $ds^2 = -\frac{|\nabla\tau|^2}{m^2}dT^2 + \frac{m^2}{|\nabla\tau|^2}dr^2$. In the second generation case, this produces a complicated expression for the nonlinear PDEs which Ψ (or Θ) must satisfy. It is likely that a more delicate treatment is necessary.

4 Three Models for Classical Gravity

4.1 The generalised JT action

In this section, we introduce a potential function $V(y)$, and use this notation to express a generalised JT-type action. The we may derive a set of equations for the dilaton, reducing to a set of integral conditions on the coordinate transformation $\Psi(T,r) = (\psi_1(T,r),\psi_2(T,r))$.

Consider the following generalised action integral

$$I[g,\tau] = \frac{1}{2G}\int_M \sqrt{|\det(g)|}dx_1dx_2(R(g)\tau + \frac{1}{l^2}V\circ\tau), \tag{21}$$

where V is an arbitrary function of the scalar field $\tau(T,r) = \frac{r}{l}$, viewed in black hole coordinates. Thus, we may view V as $V(r)$. Note that we now denote m by $\frac{1}{l}$, in order to differentiate the "mass constant" m with the above arbitrary multiplicative constant l. By

[5]note that solutions of the linearisation need not solve the dilaton equations

varying the dilaton τ and the metric g, we obtain the equations of motion

$$\begin{aligned} R(g) + \frac{1}{l^2}\frac{dV}{dx_2} \circ \tau &= 0 \\ \nabla_i \nabla_j \tau + \frac{1}{2l^2} g_{ij}(V \circ \tau) &= 0. \end{aligned} \tag{22}$$

In the previous section, we chose a metric g and solved for the dilaton τ in the equations of motion. Now, we choose $\tau(T, r) = \frac{r}{l}$ and solve the system for the metric g. In particular,

PROPOSITION 3. *Let J be defined such that $J'(y) = V(y)$. Given $f(r) = [J(r/l) + C]$, the metric g defined by $ds^2 = f(r)dT^2 - \frac{dr^2}{f(r)}$ solves the above equations of motion 22 of the dilaton $\tau(T, r) = r/l$.*

Proof. We can compute the Christoffel symbols of the claimed solution and check each equation manually. ☐

The power in this generalised action is that it allows us to work with several models of gravity at once. For the potential $V(r) = -\gamma r^\alpha$, there are three cases of interest (Gegenberg, Kunstatter, Louis-Martinez):

J-T formulation $\alpha = 1$ $\gamma = 1$: Used to form a statistical origin for black hole entropy, it is obtained by imposing axial symmetry in ordinary Einstein gravity.

String Inspired Gravity (SIG) $\alpha = 0$ $\gamma = 1$: This model for two-dimensional gravity is used to study the endpoint of gravitational collapse.

Spherically Symmetric Gravity (SSG) $\alpha = -\frac{1}{2}$ **and** $\gamma = \frac{1}{\sqrt{2}}$: By truncating all non-spherically symmetric modes in 3+1 Einstein gravity, one obtains this case.

These models are currently being studied in order to produce models for quantum gravity. Since many of the quantities under examination are "solvable" in two dimensions[6], they provide valuable theoretical insight into the behaviour of black holes. A question we are interested in addressing then, is, "can we transform these black hole metrics into a soliton metrics, and if so, what kind of soliton metrics are they?"

[6] in the absence of matter

4.2 The dilaton equations, revisited

Now we have fixed $\tau(T,r) = \frac{r}{l}$ and the metric given by $ds^2 = f(r)dT^2 - \frac{1}{f(r)}dr^2$, where $f(r) = -[J(r/l) + C]$ and $J'(y) = V(y)$. We would like to find an invertible transformation $\Psi : (T,r) \longrightarrow (x,t)$ so that the above metric transforms to the metric in 15 $ds^2 = \cos^2(\frac{u}{2})dx^2 - \sin^2(\frac{u}{2})dt^2 = \tanh^2(\rho)dx^2 - \text{sech}^2(\rho)dt^2$, where $\rho = \rho(x,t)$ is unknown (and hence so is $u = u(x,t)$). Note then, that this is not yet a soliton metric, as we do not know whether u is a soliton, or solution to a sine-Gordon equation at all! Observe however, that once we have found the transformations Ψ and Θ, then we may solve for u, and (possibly) determine what kind of equation it solves.

Note also that once we have found $\Theta(x,t)$, the dilaton equations $\nabla_i\nabla_j\tau + \frac{1}{2l^2}g_{ij}\left(-\gamma\tau^\alpha\right) = 0$ in 22 are solved in the (x,t) coordinates by the metric g and the dilaton $\tau(x,t) = \tau\circ\Theta(x,t) = \frac{\theta_2(x,t)}{l}$. In fact, given the soliton-type metric above, $\tau(x,t) = \frac{\theta_2(x,t)}{l}$ will be a solution to the system 14 with $\alpha = 1$, $\gamma = 2$ and $l = \frac{1}{m}$:

$$\begin{aligned}
\tau_{xx} + \frac{1}{2}\tan\left(\frac{u}{2}\right)u_x\tau_x + \frac{1}{2}\cot\left(\frac{u}{2}\right)u_t\tau_t - \frac{\gamma}{2l^2}\cos^2\left(\frac{u}{2}\right)\tau^\alpha &= 0\\
\tau_{tt} - \frac{1}{2}\tan\left(\frac{u}{2}\right)u_x\tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right)u_t\tau_t + \frac{\gamma}{2l^2}\sin^2\left(\frac{u}{2}\right)\tau^\alpha &= 0\\
\tau_{xt} + \frac{1}{2}\tan\left(\frac{u}{2}\right)u_t\tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right)u_x\tau_t &= 0.
\end{aligned} \tag{23}$$

In order to find Ψ, we set $x = \psi_1(T,r)$ and $t = \psi_2(T,r)$. Then $dx^2 = \left[\frac{\partial\psi_1}{\partial T}dT + \frac{\partial\psi_1}{\partial r}dr\right]^2$ and $dt^2 = \left[\frac{\partial\psi_2}{\partial T}dT + \frac{\partial\psi_2}{\partial r}dr\right]^2$. So we may rewrite our soliton metric as

$$\begin{aligned}
ds^2 &= \tanh^2(\rho)dx^2 - \text{sech}^2(\rho)dt^2\\
&= \left\{\left(\frac{\partial\psi_1}{\partial T}\right)^2 - \text{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial T}\right)^2 + \left(\frac{\partial\psi_2}{\partial T}\right)^2\right]\right\}dT^2\\
&\quad +2\left\{\left(\frac{\partial\psi_1}{\partial T}\right)\left(\frac{\partial\psi_1}{\partial r}\right) - \text{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial T}\right)\left(\frac{\partial\psi_1}{\partial r}\right) + \left(\frac{\partial\psi_2}{\partial T}\right)\left(\frac{\partial\psi_2}{\partial r}\right)\right]\right\}dTdr\\
&\quad +\left\{\left(\frac{\partial\psi_1}{\partial r}\right)^2 - \text{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial r}\right)^2 + \left(\frac{\partial\psi_2}{\partial r}\right)^2\right]\right\}dr^2\\
&= \left[J\left(\frac{r}{l}\right) + C\right]dT^2 - \frac{1}{\left[J\left(\frac{r}{l}\right) + C\right]}dr^2.
\end{aligned} \tag{24}$$

Thus Ψ must satisfy the system of nonlinear PDE equating the components of the metric:

$$\begin{aligned}\left[J\left(\frac{r}{l}\right)+C\right] &= \left\{\left(\frac{\partial\psi_1}{\partial T}\right)^2 - \text{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial T}\right)^2+\left(\frac{\partial\psi_2}{\partial T}\right)^2\right]\right\}\\ 0 &= 2\left\{\left(\frac{\partial\psi_1}{\partial T}\right)\left(\frac{\partial\psi_1}{\partial r}\right) - \text{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial T}\right)\left(\frac{\partial\psi_1}{\partial r}\right)+\left(\frac{\partial\psi_2}{\partial T}\right)\left(\frac{\partial\psi_2}{\partial r}\right)\right]\right\}\\ -\frac{1}{\left[J\left(\frac{r}{l}\right)+C\right]} &= \left\{\left(\frac{\partial\psi_1}{\partial r}\right)^2 - \text{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial r}\right)^2+\left(\frac{\partial\psi_2}{\partial r}\right)^2\right]\right\} \end{aligned} \tag{25}$$

Assume[7] now that $\frac{\partial\psi_1}{\partial T} = v$ and $\frac{\partial\psi_2}{\partial T} = 1$, where $a^2 = 1 + v^2$ so that the dT^2 coefficient becomes $v^2 - a^2\text{sech}^2(\rho\circ\Psi) \overset{set}{=} J\left(\frac{r}{l}\right) + C$. Letting $C = v^2$, then we solve for $\rho\circ\Psi$:

$$\rho\circ\Psi(T,r) = \text{arcsech}\frac{\sqrt{-J(r/l)}}{a} = \log\left[\frac{a+\sqrt{a^2+J(r/l)}}{\sqrt{-J(r/l)}}\right]. \tag{26}$$

Differentiating this expression with respect to r, we obtain

$$\rho_x\circ\Psi\left(\frac{\partial\psi_1}{\partial r}\right) + \rho_t\circ\Psi\left(\frac{\partial\psi_2}{\partial r}\right) \overset{(i)}{=} \frac{-aJ'\left(\frac{r}{l}\right)}{2lJ\left(\frac{r}{l}\right)\sqrt{a^2+J\left(\frac{r}{l}\right)}}. \tag{27}$$

Next, we set the $dTdr$ coefficient equal to zero.

$$\begin{aligned} 0 &= \left\{\left(\frac{\partial\psi_1}{\partial T}\right)\left(\frac{\partial\psi_1}{\partial r}\right) - \text{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial T}\right)\left(\frac{\partial\psi_1}{\partial r}\right)+\left(\frac{\partial\psi_2}{\partial T}\right)\left(\frac{\partial\psi_2}{\partial r}\right)\right]\right\}\\ &\overset{(ii)}{=} v\frac{\partial\psi_1}{\partial r}\left[1+\frac{J\left(\frac{r}{l}\right)}{a^2}\right] + \frac{\partial\psi_2}{\partial r}\frac{J\left(\frac{r}{l}\right)}{a^2}. \end{aligned} \tag{28}$$

Thus, by (i) and (ii) we have a linear system in the r derivatives of ψ_1 and ψ_2, which we may re-express in matrix form as

$$\begin{bmatrix} \rho_x\circ\Psi & \rho_t\circ\Psi \\ v\left(1+\frac{J\left(\frac{r}{l}\right)}{a^2}\right) & \frac{J\left(\frac{r}{l}\right)}{a^2} \end{bmatrix} \begin{bmatrix} \frac{\partial\psi_1}{\partial r} \\ \frac{\partial\psi_2}{\partial r} \end{bmatrix} = \begin{bmatrix} \frac{-aJ'\left(\frac{r}{l}\right)}{2lJ\left(\frac{r}{l}\right)\sqrt{a^2+J\left(\frac{r}{l}\right)}} \\ 0 \end{bmatrix}$$

Using Cramer's Rule, we can derive conditions on each of $\frac{\partial\psi_1}{\partial r}$ and $\frac{\partial\psi_2}{\partial r}$

$$\begin{aligned} \frac{\partial\psi_1}{\partial r} &= \frac{-J'\left(\frac{r}{l}\right)}{2al\left[v^2+J\left(\frac{r}{l}\right)\right]\sqrt{a^2+J\left(\frac{r}{l}\right)}(\rho_x\circ\Psi)}\\ \frac{\partial\psi_2}{\partial r} &= \frac{vJ'\left(\frac{r}{l}\right)\sqrt{a^2+J\left(\frac{r}{l}\right)}}{2al\left[v^2+J\left(\frac{r}{l}\right)\right]J\left(\frac{r}{l}\right)(\rho_x\circ\Psi)}. \end{aligned} \tag{29}$$

[7] These assumptions were made to correspond to Williams' results

Finally, we use these two equations in the third coefficient equation, the dr^2 equation $\left(\frac{\partial\psi_1}{\partial r}\right)^2 - \mathrm{sech}^2(\rho\circ\Psi)\left[\left(\frac{\partial\psi_1}{\partial r}\right)^2 + \left(\frac{\partial\psi_2}{\partial r}\right)^2\right] \overset{set}{=} \frac{-1}{J\left(\frac{r}{l}\right)+C}$, to write $(\rho_x\circ\Psi)^2 = \frac{-J'\left(\frac{r}{l}\right)^2}{4a^2l^2J\left(\frac{r}{l}\right)}$. Thus, we can rewrite the equations for ψ_1, and ψ_2 independently of the unknown $\rho\circ\Psi$:

$$\begin{aligned}\frac{\partial\psi_1}{\partial r} &= \frac{\sqrt{-J\left(\frac{r}{l}\right)}}{\left[v^2+J\left(\frac{r}{l}\right)\right]\sqrt{a^2+J\left(\frac{r}{l}\right)}} \\ \frac{\partial\psi_2}{\partial r} &= \frac{v\sqrt{a^2+J\left(\frac{r}{l}\right)}}{\left[v^2+J\left(\frac{r}{l}\right)\right]\sqrt{-J\left(\frac{r}{l}\right)}}.\end{aligned} \tag{30}$$

In practice, $J(x) = -\gamma x^\alpha$, so that the solutions are real-valued. If we are able to integrate these equations with respect to r, then our initial assumption that $\frac{\partial\psi_1}{\partial T} = v$ and $\frac{\partial\psi_2}{\partial T} = 1$, will solve the T variable and we will have found our transformation Ψ between the soliton metric and the black hole metric.

4.3 Application to three classical models of gravity

In the case where we choose a specific potential function (for each of the three models for classical gravity), we summarise our results in a proposition.

Proposition 4. *Consider the generalised black hole metric given by* $ds^2 = f(r)dT^2 - \frac{1}{f(r)}dr^2$ *in Proposition 3, with* $V(r) = -\gamma r^\alpha$, $C = v^2$ *and* $a^2 = 1+v^2$. *Choose the dilaton* $\tau(T,r) = \frac{r}{l}$. *Then in each of the Jackiw-Teitelboim (JT) and String Inspired (SIG) models for gravity, there exists an invertible coordinate transformation* Ψ, *sending the black hole metric above to the generalised soliton metric found in 15* $ds^2 = \tanh^2(\rho(x,t))dx^2 - sech^2(\rho(x,t))dt^2 = \cos^2(\frac{u}{2})dx^2 - \sin^2(\frac{u}{2})dt^2$. *Such a map satisfies the equations*

$$\begin{aligned}\psi_1(T,r) &= vT + \int \frac{\sqrt{-J\left(\frac{r}{l}\right)}}{\left[v^2+J\left(\frac{r}{l}\right)\right]\sqrt{a^2+J\left(\frac{r}{l}\right)}}dr \\ \psi_2(T,r) &= T + \int \frac{v\sqrt{a^2+J\left(\frac{r}{l}\right)}}{\left[v^2+J\left(\frac{r}{l}\right)\right]\sqrt{-J\left(\frac{r}{l}\right)}}dr\,.\end{aligned} \tag{31}$$

Conjecture 4.1. Under the same hypotheses as above, the results of the proposition also hold for the Spherically Symmetric Gravity model (SSG).

These are three classical models for which we consider computation of the Ψ integrals.

EXAMPLE 2. In the JT case, the potential function appearing in the generalised action functional is given by $V(r) = -\gamma r$, so we obtain $J\left(\frac{r}{l}\right) = -\frac{\gamma}{2l^2}r^2$ and the integrals under consideration are of the form

$$\begin{aligned}\psi_1(T,r) &= vT + C_1 \int \frac{r}{[v^2 - B^2r^2]\sqrt{a^2 - B^2r^2}}dr \\ \psi_2(T,r) &= T + C_2 \int \frac{\sqrt{a^2 - B^2r^2}}{[v^2 - B^2r^2]\, r}dr, \end{aligned} \tag{32}$$

where C_1, C_2, B are constants in terms of γ, l, a, and v. We can integrate this exactly (using MAPLE or by hand) to produce precisely the Ψ stated in Theorem 1. Inverting this Ψ, of course, gives us the Θ previously claimed as well. Note that in the case $\alpha = 1$, we have reduced the generalised dilaton equations 23 to precisely those in section 3 (see equation 14). Note also that $\tau \circ \Theta(x,t) = \frac{\theta_2(x,t)}{l} = a\text{sech}(\rho(x,t))$ agrees precisely with the result previously stated.

EXAMPLE 3. In the String Inspried case (SIG), where the potential is given by $V(r) = -\gamma \Rightarrow J\left(\frac{r}{l}\right) = -\gamma r$, one expects simpler integrands.

$$\begin{aligned}\psi_1(T,r) &= vT + C_1 \int \frac{\sqrt{r}}{[v^2 - B^2r]\sqrt{a^2 - B^2r}}dr \\ \psi_2(T,r) &= T + C_2 \int \frac{\sqrt{a^2 - B^2r}}{[v^2 - B^2r]\sqrt{r}}dr, \end{aligned} \tag{33}$$

from which we produce a transformation Ψ, where $w = v^2 - \frac{\gamma}{l}r$:

$$\begin{aligned}\psi_1(T,r) &= vT - \frac{l}{\gamma}\arcsin\left(\frac{v^2 - 2w - 1}{a^2}\right) + \frac{lv}{\gamma}\text{arctanh}\left(\frac{2v^2 + v^2w - w}{2v\sqrt{(v^2 - w)(1 + w)}}\right) \\ \psi_2(T,r) &= \frac{\psi_1(T,r)}{v} + \frac{a^2 l}{v\gamma}\arcsin\left(\frac{v^2 - 2w - 1}{a^2}\right) \end{aligned} \tag{34}$$

It turns out that it is possible, to invert this transformation to produce also a map Θ defined by

$$\begin{aligned}\theta_1(x,t) &= \frac{1}{a^2}\left\{vx + t - \frac{la^2}{\gamma}\text{arctanh}\left(\frac{a^2 + (v^2 - 1)\sin(\frac{(x - vt)\gamma}{la^2})}{2v\cos(\frac{(x - vt)\gamma}{la^2})}\right)\right\} \\ \theta_2(x,t) &= \frac{1}{2}\left\{v^2 - 1 + a^2\sin(\frac{(x - vt)\gamma}{la^2})\right\} \end{aligned} \tag{35}$$

Note that the dilaton in soliton coordinates is again $\tau(x,t) = \frac{\theta_2(x,t)}{l}$. Observe that we must also specify $\rho(x,t)$ so that we may determine the kind of soliton metric to which we are transforming. In particular,

$$\rho(x,t) = \text{arcsech}\left(\sqrt{\frac{1-\sin(\frac{(x-vt)\gamma}{la^2})}{2}}\right),$$

and our soliton $u = 4\arctan(e^{\rho(x,t)})$ is in fact harmonic-i.e. satisfies the Sine-Gordon equation for $m^2 = 0$.

EXAMPLE 4. Finally, in the Spherically Symmetric model (SSG), the potential is $V(r) = -\frac{1}{\sqrt{2r}}$ giving us $J\left(\frac{r}{l}\right) = -\sqrt{\frac{2\gamma}{l}}\sqrt{r}$. In this case, we have a set of integrals which still are currently being explored

$$\begin{aligned} \psi_1(T,r) &= vT + C_1\int \frac{\sqrt{\sqrt{r}}}{[v^2 - B^2\sqrt{r}]\sqrt{a^2 - B^2\sqrt{r}}}dr \\ \psi_2(T,r) &= T + C_2\int \frac{\sqrt{a^2 - B^2\sqrt{r}}}{[v^2 - B^2\sqrt{r}]\sqrt{\sqrt{r}}}dr. \end{aligned} \tag{36}$$

The above inegrals have not been successfully computed directly using a computer algebra system, so another technique may be necessary to fully address this scenario.

5 Conclusions, Conjectures and Speculations

One reason the final proposition fails in helping us to find Ψ in the breather case for the JT model is that the gauge choice $\tau(T,r) = \frac{r}{l}$ does not appear to be compatible with the already computed dilaton $\tau(x,t)$ in soliton coordinates. Fortunately, an alternate formulation of the target metric is known in terms of $|\nabla\tau|$. This formulation was found by studying the Killing equations of the spacetime. Since constant curvature spaces are maximally symmetric, in two dimensions there are three Killing vectors. It is possible that further study of the relationship between the dilaton and these vectors will yield more information about the two metrics involved. Future work could involve studing geodesics near the event horizon of a black hole, or comparing geodesics of different solutions $u(x,t)$.

The two dimensional JT model has been a “useful theoretical laboratory” in which many physical quantities can be explicitly computed, both in the classical and quantum cases. Despite the dimension reduction, these two dimensional models appear to still exhibit some of the usual thermodynamic properties of black holes (Gegenberg, Kunstatter). Furthermore, black hole solutions to JT gravity are related directly to a current 2+1 dimensional model: they are two-dimensional slices of the BTZ black hole, a model which has been of much interest in recent years (Banados, Teitelboim, Zanelli). It appears that the "toy model" may still provide insight into Einstein gravity and string theory yet.

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