Classical Strings and Solitons

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Abstract

In the theme of the last two lectures, we discuss solutions of two-dimensional string theory in the Schwarzschild and light-cone gauges. We shall also explore the relationship of the string action with a generalisation of the Jackiw-Teitelboim action. Using the Schwarzschild and sine-Gordon gauges in the new "JT context," we find new solutions and discuss their connection with the original string theory.

1 Introduction

1.1 Background

Let (M, g) be a pseudo-Riemannian manifold with $\dim(M) = n$ and suppose g is given locally by $ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$.

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{l=1}^{n} g^{lk} \left[\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right] \tag{1}$$

$$R_{ijk}^{l} = \frac{\partial \Gamma_{jk}^{l}}{\partial x_i} + \frac{\partial \Gamma_{ik}^{l}}{\partial x_j} + \sum_{p=1}^{n} [\Gamma_{ip}^{l} \Gamma_{jk}^{p} - \Gamma_{jp}^{l} \Gamma_{ik}^{p}] \tag{2}$$

$$R_{ij} = \sum_{l=1}^{n} R_{ilj}^{l} \tag{3}$$

$$-2K = R = \sum_{i,j=1}^{n} g^{ij} R_{ij} \tag{4}$$

$$\begin{aligned} \nabla_i \nabla_j h &= -\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial h}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\frac{\partial h}{\partial x_i} \right) \qquad (5) \\ &= -\sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial h}{\partial x_k} + \frac{\partial^2 h}{\partial x_i \partial x_j} \end{aligned}$$

1.2 Two-Dimensional String Theory: the general idea

Consider d scalar fields x^i coupled to two-dimensional gravity. In the conformal gauge, Mandal, et. al. state there is the $(d+1)$-dimensional "graviton"-like operator

$$\int d^2\xi \sqrt{\hat{g}}\hat{g}^{\alpha\beta} h_{\mu\nu}(x^i,\eta)\partial_\alpha x^\mu \partial_\beta x^\nu, \tag{6}$$

where $\mu,\nu = 1,2,\ldots,d+1$, $x^\mu = (x^i,\eta)$, $i = 1,2,\ldots,d$, and $Q = \sqrt{(25-d)/3}$ is the background charge, related to a central charge C by $-C = Q^2$.

By using a perturbations, it is claimed that "Two-dimensional gravity coupled to d scalar fields can be regarded as a critial string theory in $D = d+1$ target space dimensions [1]."

2 The Target Space Action

Consider the 'target space action" given by

$$S(g, \Phi, T) = \int d^2x e^{-2\Phi} \sqrt{|\det(g)|} \left[R(g) - 4|\nabla\Phi|^2 + |\nabla T|^2 + V(T) \right], \tag{7}$$

where g is a two-dimensional metric, Φ and T are scalar fields, known as the dilaton field and the tachyon field, respectively, and V is a potential function.

The resulting field equations are given by $(1 \leq i, j \leq 2)$

$$\begin{aligned} R_{ij} - 2\nabla_i\nabla_j\Phi + \nabla_i T \nabla_j T &= 0 \\ R + 4|\Phi|^2 - 4\nabla^2\Phi + |\nabla T|^2 + V(T) + C &= 0 \\ -2\nabla^2 T + 4\nabla\Phi\nabla T + V'(T) &= 0. \end{aligned} \tag{8}$$

Assume $T = 0$ so that the equations reduce to the following system

$$R_{ij} - 2\nabla_i\nabla_j\Phi = 0 \tag{9}$$

$$R + 4|\Phi|^2 - 4\nabla^2\Phi + C = 0 \tag{10}$$

We will solve this simpler system explicitly in two different gauges, i.e. using two different simplifications.

2.1 A Schwarzschild Gauge

Let $(x_1, x_2) = (t, \eta)$, and assume that the metric g is of the form

$$ds^2 = -g(\eta)dt^2 + \frac{d\eta^2}{g(\eta)},$$

so that g is of classical black hole type. Suppose also that

$$\Phi(t, \eta) = K\eta,$$

for some constant K.

In two dimensions, recall $R_{ij} = \frac{1}{2}R(g)g_{ij}$, and computing the components of the Hessian is straightforward, so we may easily rewrite the first three field equations (9):

$$\begin{aligned} -\frac{1}{2}g''(\eta)g(\eta) + Kg'(\eta)g(\eta) &= 0 \\ 0 - 2(0) &= 0 \\ \frac{1}{2}\frac{g''(\eta)}{g(\eta)} - K\frac{g'(\eta)}{g(\eta)} &= 0. \end{aligned} \tag{11}$$

We may also compute $|\nabla\Phi|^2$ and $\nabla^2\Phi = \triangle\Phi$ =Laplace-Beltrami operator of g fairly easily to obtain the final field equation

$$g''(\eta) - 4Kg'(\eta) + 4K^2g + C = 0. \tag{12}$$

By solving the homogeneous counterpart of (12) to have solution $g_H(\eta) = c_1 e^{2K\eta} + c_2 \eta e^{2K\eta}$ and observing that a solution to the inhomogeneous equation is $g_I(\eta) = \frac{-C}{4K^2}$, we have found a general solution to the final field equation.

Thus, choosing K such that $4K^2 = -C$ (i.e. $K = \frac{Q}{2}$), $c_2 = 0$ and $c_1 = -a \in \mathbb{R}$, we solve also the first three field equations, appearing in (11).

To summarize, we have found the metric-dilaton pair solving the field equations in the Schwarzschild gauge to be

$$\begin{aligned} ds^2 &= -\left(1 - ae^{Q\eta}\right) dt^2 + \frac{d\eta^2}{(1 - ae^{Q\eta})} \\ \Phi(\eta) &= \frac{Q}{2}\eta \qquad (Q^2 = -C). \end{aligned} \tag{13}$$

We make the a few observations about the nature of this solution

1. For positive a, $g(\eta)$ has a zero at $\eta = \frac{-1}{Q} \ln a$. This indicates the presence of a horizon here, just like the Schwarzschild solution of 4-dimensional General Relativity. Also, a is related to the mass of the black hole.

2. The scalar curvature $R = g''(\eta) = -aQ^2 e^{Q\eta}$ has curvature singularity for $\eta \to +\infty$ if $Q > 0$ and for $\eta \to -\infty$ if $Q < 0$.

2.2 A Conformal Light-Cone Gauge

Next, start with $(x_1, x_2) = (x, y)$ and assume a conformal gauge $ds^2 = e^\sigma(dx^2 - dy^2)$, where $\sigma = \sigma(x, y)$. By setting $u = x + y$ and $v = x - y$, we change to light-cone coordinates

$$ds^2 = e^{\sigma(u,v)} du dv. \tag{14}$$

Then the field equations (9) can again easily be re-expressed as

$$\begin{aligned} 0 + 2\left(\partial_u\sigma\partial_u\Phi + \partial_u\partial_u\Phi\right) &= 0 \\ \partial_u\sigma\partial_v\sigma - 2\left(0 + \partial_u\partial_v\Phi\right) &= 0 \\ 0 + 2\left(\partial_v\sigma\partial_v\Phi + \partial_v\partial_v\Phi\right) &= 0 \\ 4e^{-\sigma}\left[\partial_u\sigma\partial_v\sigma + 4\partial_u\Phi\partial_v\Phi - 4\partial_u\partial_v\Phi\right] + C &= 0, \end{aligned} \tag{15}$$

where $\partial_u = \frac{\partial}{\partial u}$.

Examining the second equation in (15) above $\partial_u\partial_v(\sigma - 2\Phi) = 0$, we see that $\sigma - 2\Phi = F(u) + G(v)$, for some functions F, G. Suppose that $F = G = 0$ identically, to simplify the expression

$$\sigma = 2\Phi. \tag{16}$$

Then we need only find σ in order to have solved the field equations!

By using the first and third equations in (15), we find that

$$\frac{1}{2}e^{-2\Phi} = Auv + Bu + Cv + D,$$

which after a shift of u, v by constants yields

$$e^{-2\Phi} = Muv + N. \tag{17}$$

Upon checking the final field equation, we see that $M = \frac{-C}{4} = \frac{Q^2}{4}$ and $N = a \in \mathbb{R}$ is a free parameter.

To summarize, we have found the metric-dilaton pair solving the field equations in the Light-Cone gauge to be

$$\begin{aligned} ds^2 &= \frac{dudv}{\frac{Q^2}{4}uv + a} \\ \Phi(u, v) &= \frac{-1}{2}\ln(\frac{Q^2}{4}uv + a). \end{aligned} \tag{18}$$

We make the a few observations about the nature of this solution

1. The metric appears to be a black hole in Kruskal coordinates with horizon given by lines $uv = 0$.

2. The solution has scalar curvature $R = 4e^{-\sigma}\partial_u\partial_v\sigma = \frac{\frac{-Q^2}{4}a}{\frac{Q^2}{4}uv+a}$, and hence curvature singularity when $uv = \frac{-4a}{Q^2}$. Note that this is also a coordinate singularity for the metric.

It is possible to write down the explicit transformation between the Schwarzschild metric and the light-cone metric, given by

$$
\begin{aligned}
u &= e^{\frac{-Q}{2}(\eta'+t)} \\
v &= -\epsilon e^{\frac{-Q}{2}(\eta'-t)},
\end{aligned}
$$

where $\frac{Q^2}{16}e^{Q\eta'} = \frac{Q\eta}{1-ae^{Q\eta}}$ and ϵ is +1 inside the horizon and -1 outside the horizon. [1] claims that the (u, v)-space thus provides a 2 fold cover of the (t, η)-space.

3 Transforming the String Action

Let us return to the action yielding the field equations we have just solved

$$S(\hat{g}, \Phi) = \int d^2x e^{-2\Phi} \sqrt{|\det(\hat{g})|} \left[R(\hat{g}) - 4|\nabla\Phi|^2 + \beta \right]. \qquad (19)$$

Observe that under a conformal transformation $\hat{g} = e^{2\Phi} g$, one may rewrite the action integral as

$$I(g, \tau) = \int d^2x \sqrt{|\det(g)|} \left[R(g)\tau + \beta \right], \qquad (20)$$

where $\tau = e^{-2\Phi}$. We note that this is a special case of the generalised JT Action, found in [2], [3]

$$I(g, \tau) = \int d^2x \sqrt{|\det(g)|} \left[R(g)\tau + mV \circ \tau \right], \qquad (21)$$

where $V(x) = -\gamma x^{\alpha}$ is a potential function. When $\alpha = 1$, this action reduces to the classical Jackiw-Teitelboim model for two-dimensional gravity (JT); in the case $\alpha = 0$, we have the String Inspired Model (SIG), and for $\alpha = \frac{-1}{2}$, it is the Spherically Symmetric Theory (SSG).

3.1 The New Field Equations

The generalised JT action has field equations given by

$$\begin{aligned} R + m^2 \frac{dV}{dx} \circ \tau &= 0 \\ \nabla_i \nabla_j \tau + \frac{m^2}{2} g_{ij} V \circ \tau &= 0. \end{aligned} \tag{22}$$

Just as before, one may choose various gauges in which to work to solve these equations. We shall consider two coordinate systems: a Schwarzschild-type gauge and a sine-Gordon gauge.

3.2 The Schwarzschild Gauge, Revisited

Let $(x_1, x_2) = (T, r)$ and assume that the metric g is of the form

$$ds^2 = -g(r)dT^2 + \frac{dr^2}{g(r)},$$

so that g is of classical black hole type. Suppose also that

$$\tau(T, r) = mr,$$

so that the dilaton equations become

$$\begin{aligned} R = g''(r) &= 0 \\ -\frac{m}{2}g(r)g'(r) + \frac{m^2\gamma}{2}g(r) &= 0 \\ \nabla_1\nabla_2\tau + g_{12}(\ldots) = 0 &= 0 \\ \frac{mg'(r)}{2g(r)} + \frac{-m^2\gamma}{2g(r)} &= 0. \end{aligned} \tag{23}$$

Then one can verify that the following metric-dilaton pair solve the field equations

$$\begin{aligned} ds^2 &= (J(mr) + C)dT^2 - \frac{dr^2}{(J(mr) + C)} \\ &= (-m\gamma r + C)\, dT^2 - \frac{dr^2}{(-m\gamma r + C)} \\ \tau(T, r) &= mr, \end{aligned} \tag{24}$$

where $J'(x) = V(x) = \gamma\, x$.

Of course we see that the expression has a coordinate singularity for $r = \frac{C}{m\gamma}$, but no curvature singularities at all.

3.3 The sine-Gordon Gauge

Finally, choose $(x_1, x_2) = (x, t)$ and assume a sine-Gordon type metric

$$ds^2 = \cos^2\left(\frac{u}{2}\right) dx^2 - \sin^2\left(\frac{u}{2}\right) dt^2, \tag{25}$$

where $u = u(x,t)$ must be a solution of the Euclidean sine-Gordon equation $u_{xx} + u_{tt} = m^2 \sin(u)$ in order for the metric to satisfy the first field equation in (22). Then, either one must solve the remaining field equations

$$\begin{aligned}
0 &= \tau_{xx} + \frac{1}{2}\tan\left(\frac{u}{2}\right) u_x \tau_x + \frac{1}{2}\cot\left(\frac{u}{2}\right) u_t \tau_t - m^2 \cos^2\left(\frac{u}{2}\right) \tau \\
0 &= \tau_{tt} - \frac{1}{2}\tan\left(\frac{u}{2}\right) u_x \tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right) u_t \tau_t + m^2 \sin^2\left(\frac{u}{2}\right) \tau \\
0 &= \tau_{xt} + \frac{1}{2}\tan\left(\frac{u}{2}\right) u_t \tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right) u_x \tau_t,
\end{aligned} \tag{26}$$

or transform the existing Schwarzschild solution to the sine-Gordon space. It turns out that this is a slightly easier task.

Starting with the black hole metric given in (24), we search for a transformation $\Psi(T,r) = (\psi_1(T,r), \psi_2(T,r))$ sending this metric to the sine-Gordon metric above. The resulting coupled nonlinear partial differential equations that ψ_1 and ψ_2 must satisfy can be reduced using some simplifying assumptions. The task becomes one of solving the following decoupled system of ODE:

$$\begin{aligned} \frac{\partial \psi_1}{\partial r} &= \frac{\sqrt{-J\,(mr)}}{\left[v^2 + J\,(mr)\right]\sqrt{a^2 + J\,(mr)}} \\ \frac{\partial \psi_2}{\partial r} &= \frac{v\sqrt{a^2 + J\,(mr)}}{\left[v^2 + J\,(mr)\right]\sqrt{-J\,(mr)}}. \end{aligned} \tag{27}$$

In the case where $V(x) = -\gamma \Rightarrow J(x) = \gamma x$, one obtains the map Ψ and the inverse $\Theta = \Psi^{-1}$:

$$\begin{aligned} \theta_1(x,t) &= \frac{1}{a^2}\left\{ vx + t - \frac{la^2}{\gamma}\text{arctanh}\left(\frac{a^2 + (v^2-1)\sin(\frac{(x-vt)\gamma}{la^2})}{2v\cos(\frac{(x-vt)\gamma}{la^2})} \right) \right\} \\ \theta_2(x,t) &= \frac{1}{2}\left\{ v^2 - 1 + a^2 \sin(\frac{(x-vt)\gamma}{la^2}) \right\}, \end{aligned} \tag{28}$$

where $a^2 = 1 + v^2$.

To summarize, the metric-dilaton pair in the sine-Gordon gauge solving the field equations in (22) is given by

$$\begin{aligned} ds^2 &= \cos^2\left(\frac{u}{2}\right)dx^2 - \sin^2\left(\frac{u}{2}\right)dt^2 \\ u(x,t) &= 4\arctan\left\{ exp\left(\text{arcsech}\left[\sqrt{\frac{1-\sin(\frac{(x-vt)\gamma}{la^2})}{2}}\right]\right)\right\} \\ \tau(x,t) &= \theta_2(x,t)/l, \end{aligned} \tag{29}$$

where we note that u is in fact a harmonic function, satisfying

$$u_{xx} + u_{tt} = m^2 \sin u,$$

for $m = 0$.

Again, there are no curvature singularities, as $R = 0$ (note that the domain of $arcsech(y)$ is $0 < y \leq 1$).

4 Further Exploration

We have explicated the map between the one-parameter family of Schwarzschild-type solutions and light-cone solutions in the original String Action, in addition to the map between Schwarzschild-type solutions and sine-Gordon metric solutions in the String Inspired action. One avenue currently being explored involved establishing a dictionary of transformations between the two action integral perspectives in order to better understand the roles of solitons in string theory.

References

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