

Solving a complicated eigenvalue problem using a simple black hole

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Abstract

In recent years, the soliton metric given by $ds^2 = \cos(w)^2 dx^2 - \sin(w)^2 dt^2$ has been of particular interest to physicists studying the Jackiw-Teitelboim model for gravity [2]. In this talk, I will provide a transformation from such a metric to a two-dimensional slice of the Banados-Teitelboim-Zanelli (BTZ) black hole metric. In fact, this transformation will allow us to reduce a difficult, highly nonlinear eigenvalue problem to a slightly simpler one which is solved by using hypergeometric functions.

1 Introduction and Background

Let (M, g) be a pseudo-Riemannian manifold with $\dim(M) = n$ and suppose g is given locally by $ds^2 = \sum_{i,j=1}^n g_{ij} dx_i dx_j$.

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^n g^{lk} \left[\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right] = \text{Christoffel Symbols} \quad (1)$$

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x_i} + \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_{p=1}^n [\Gamma_{ip}^l \Gamma_{jk}^p - \Gamma_{jp}^l \Gamma_{ik}^p] = \text{Curvature Tensor} \quad (2)$$

$$R_{ij} = \sum_{l=1}^n R_{ilj}^l = \text{Ricci Tensor} \quad (3)$$

$$-2K = R = \sum_{i,j=1}^n g^{ij} R_{ij} = \text{Scalar Curvature} \quad (4)$$

$$\begin{aligned} \nabla_i \nabla_j h &= -\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial h}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\frac{\partial h}{\partial x_i} \right) \\ &= -\sum_{k=1}^n \Gamma_{ij}^k \frac{\partial h}{\partial x_k} + \frac{\partial^2 h}{\partial x_i \partial x_j} = \text{Hessian} \end{aligned} \quad (5)$$

2 The Einstein Equations and the JT Action

Consider the Einstein vacuum equations in two dimensions

$$R_{ij} - \frac{R}{2}g_{ij} + \Lambda g_{ij} = K^2 T_{ij} = 0.$$

Observe that when $n = 2$, we have $R_{ij} = (R/2)g_{ij}$, so the vacuum ($T_{ij} = 0$) equations are trivially satisfied for $\Lambda = 0$.

Thus, in two dimensional spacetime, Jackiw and Teitelboim constructed a nontrivial theory of gravity for $T_{ij} = 0$ via the following (JT) action

$$I_{JT}[\tau, g] = \frac{1}{2G} \int_M \sqrt{|\det(g)|} dx_1 dx_2 (A - R(g)) \tau(x_1, x_2), \quad (6)$$

where $\tau = \tau(x_1, x_2)$ is a scalar field [5, 6]. Upon varying the dilaton τ and the metric g one obtains the field equations

$$\begin{cases} R - \frac{A}{\tau} & = 0 \\ (\nabla_i \nabla_j - A g_{ij}) \tau & = 0 \end{cases} \quad (7)$$

To solve the field equations in (7), we search for a pair (g, τ) such that $R = R(g)$ is constant and τ satisfies the resulting field equations for the given metric.

3 Solitons and sine-Gordon equations

One useful candidate for g is a metric of the form

$$ds_{sol}^2 = \cos^2\left(\frac{u}{2}\right) dx^2 - \sin^2\left(\frac{u}{2}\right) dt^2, \quad (8)$$

where $u = u(x_1, x_2) = u(x, t)$.

Observe that for this metric,

$$R(g) = \frac{2(u_{xx} + u_{tt})}{\sin(u)} = \frac{2\Delta u}{\sin(u)},$$

so $R(g) = A \iff \Delta u = u_{xx} + u_{tt} = \frac{A}{2} \sin(u)$. That is, $u(x, t)$ must satisfy the *Euclidean sine-Gordon equation*.

Solutions of this equation are called *solitons* and hence g is referred to as a *soliton metric*. A soliton is a localised large amplitude solid wave which propagates without spreading. It has particle-like, nondispersive properties: solitons can pass through each other and preserve their speed and shape after collision.

kink, antikink (soliton, antisoliton): $u(x, t) = 4 \arctan(e^{\pm\rho(x,t)})$, where $\rho(x, t) = \frac{m}{a}(x - vt)$, and $a^2 = 1 + v^2$.

oscillating kink-antikink: $u(x, t) = 4 \arctan\left(\frac{v \sinh(amx)}{a \cos(vmt)}\right)$, $a^2 = 1 + v^2$.

breather: $u(x, t) = 4 \arctan\left(\frac{a \sin(vmt)}{v \cosh(amx)}\right)$, where $a^2 = 1 - v^2$. Note this is a solution of the *standard* sine-Gordon equation $\square u = u_{xx} - u_{tt} = m^2 \sin(u)$.

4 Black holes and the Schwarzschild metric

Another useful candidate to consider for g when searching for the solution pair (g, τ) is the *black hole metric* given by

$$ds_{bh}^2 = - \left(\lambda r^2 - M + \frac{J^2}{4r^2} \right) dT^2 + \left(\lambda r^2 - M + \frac{J^2}{4r^2} \right)^{-1} dr^2, \quad (9)$$

where the coordinates in (7) are given by $(x_1, x_2) = (T, r)$.

This metric is modeled after non-angular terms of the classical Schwarzschild metric

$$ds^2 = \left(1 - \frac{2Gm}{c^2 r} \right)^{-1} dr^2 - c^2 \left(1 - \frac{2Gm}{c^2 r} \right) dT^2 + r^2 d\theta^2 + r^2 \sin^2 \varphi d\varphi^2.$$

Note that the two-dimensional black hole metric in (9) has scalar curvature $R = 2\lambda + \frac{3J^2}{2r^4}$, and thus $R = 2m^2$ if and only if the angular momentum J vanishes.

5 Some natural questions arise

1. Is it possible to find τ given either metric g ?
2. Can we find explicitly transformations between ds_{sol}^2 and ds_{bh}^2 in order to change one pair of solutions to the other, namely, $(g(x, t), \tau(x, t)) \longrightarrow (g(T, r), \tau(T, r))$? If so, is it possible to find a general method for constructing maps between soliton metrics and black hole metrics?
3. How do problems in one metric translate to problems in the other metric¹?

¹For instance, compare coordinate singularities of the two metrics.

6 Laplace-Beltrami Operator

Recall the Laplace-Beltrami operator of a diagonal metric g is defined to be

$$\Delta_g \stackrel{def}{=} \nabla_g^2 := \frac{1}{\sqrt{|\det g|}} \sum_{j=1}^2 \frac{\partial}{\partial x_j} \left(\sqrt{|\det g|} g^{jj} \frac{\partial}{\partial x_j} \right). \quad (10)$$

EXAMPLE 1. Consider the generalised black hole metric given by $ds^2 = A(r)dT^2 - \frac{1}{A(r)}dr^2$. Then the Laplacian of this metric is easily computed to be

$$\nabla_{bh}^2 = \frac{1}{A(r)} \frac{\partial^2}{\partial T^2} - A(r) \frac{\partial^2}{\partial r^2} - A'(r) \frac{\partial}{\partial r}. \quad (11)$$

EXAMPLE 2. Consider the soliton metric given by

$$ds^2 = \cos^2\left(\frac{u}{2}\right) dx^2 - \sin^2\left(\frac{u}{2}\right) dt^2.$$

Then in an appropriate domain, the Laplacian of this metric found to be

$$\nabla_{sol}^2 = \frac{u_x/2}{\cos^3\left(\frac{u}{2}\right) \sin\left(\frac{u}{2}\right)} \frac{\partial}{\partial x} + \frac{1}{\cos^2\left(\frac{u}{2}\right)} \frac{\partial^2}{\partial x^2} + \frac{u_t/2}{\cos\left(\frac{u}{2}\right) \sin^3\left(\frac{u}{2}\right)} \frac{\partial}{\partial t} - \frac{1}{\sin^2\left(\frac{u}{2}\right)} \frac{\partial^2}{\partial t^2}. \quad (12)$$

7 The Conjecture

Our goal in this talk is to understand better the relation between SGE and black holes in the following context:

CONJECTURE 1. *Let $\nu \in \mathbb{R}$ and let g_{sol} be any soliton metric. Then there exists a nontrivial solution $F = F(x, t)$ of the eigenvalue problem*

$$\Delta_{sol} F = \nu F. \tag{13}$$

We will use Δ_{sol} and Δ_{bh} to refer to the Laplace-Beltrami operators of the soliton metric ds_{sol}^2 and black hole metric ds_{bh}^2 appearing in (8) and (9), respectively.

To solve Conjecture (1), we will employ a general gameplan involving five primary steps.

1. Examine a known result relating a specific soliton eigenvalue problem to a specific black hole eigenvalue problem.
2. Compute a new, explicit example to test the known result, i.e. see if a new set of soliton coordinates can be rewritten precisely as black hole coordinates.
3. Specialise the Conjecture, or generalise the known result to include this new example, if possible.
4. Reformulate the Eigenvalue problem in terms of black holes
5. Solve the new black hole eigenvalue problem!

8 A small note on the Dilaton Equations

We start by letting $A = 2m^2$ and denote (x_1, x_2) by (x, t) so that we may restrict our attention to the soliton metric g as in (8), given by $ds^2 = \cos^2\left(\frac{u}{2}\right) dx^2 - \sin^2\left(\frac{u}{2}\right) dt^2$, with $u = u(x, t)$ satisfying $\Delta u = u_{xx} + u_{tt} = m^2 \sin(u)$. Then by direct computation, we have (again) that $R = 2m^2 = A$.

For this choice of metric, the remaining equations of the JT action, known as equations of motion for the dilaton $\tau(x, t)$ given in (7) reduce to

$$\tau_{xx} + \frac{1}{2} \tan\left(\frac{u}{2}\right) u_x \tau_x + \frac{1}{2} \cot\left(\frac{u}{2}\right) u_t \tau_t - m^2 \cos^2\left(\frac{u}{2}\right) \tau = 0 \quad (14)$$

$$\tau_{tt} - \frac{1}{2} \tan\left(\frac{u}{2}\right) u_x \tau_x - \frac{1}{2} \cot\left(\frac{u}{2}\right) u_t \tau_t + m^2 \sin^2\left(\frac{u}{2}\right) \tau = 0 \quad (15)$$

$$\tau_{xt} + \frac{1}{2} \tan\left(\frac{u}{2}\right) u_t \tau_x - \frac{1}{2} \cot\left(\frac{u}{2}\right) u_x \tau_t = 0. \quad (16)$$

Thus, for a given choice of $u(x, t)$, we may attempt to find a dilaton and thus produce the solution (g, τ) .

9 An example: the kink soliton

THEOREM 2. (*Williams [1]*) Let $u = u(x, t) = 4 \arctan(e^{\rho(x,t)})$, where $\rho(x, t) = \frac{m}{a}(x - vt)$, $a^2 = 1 + v^2$ be the kink soliton solution of the Euclidean sine-Gordon equation. Construct the soliton metric

$$ds^2 = \cos^2\left(\frac{u}{2}\right) dx^2 - \sin^2\left(\frac{u}{2}\right) dt^2 = \tanh^2(\rho) dx^2 - \operatorname{sech}^2(\rho) dt^2. \quad (17)$$

For τ defined by $\tau(x, t) = a \operatorname{sech}(\rho(x, t))$, the pair (g, τ) solve the dilaton equations (14).

Moreover, there exists a coordinate transformation $\Psi(T, r) = (\psi_1, \psi_2)$ taking the soliton metric (17) above to the black hole metric in (9) having $J = 0$ and $\lambda = m^2$ given by

$$ds^2 = (M - m^2 r^2) dT^2 - \frac{1}{(M - m^2 r^2)} dr^2, \quad \text{for } M = v^2. \quad (18)$$

Proof. Let $\Psi(T, r) = (\psi_1(T, r), \psi_2(T, r))$ be defined by

$$x = \psi_1(T, r) = vT + \frac{1}{2m} \log \left[\frac{\sqrt{a^2 - m^2 r^2} + 1}{\sqrt{a^2 - m^2 r^2} - 1} \right] \quad (19)$$

$$t = \psi_2(T, r) = \frac{\psi_1}{v} - \frac{a}{mv} \log \left[\frac{a + \sqrt{a^2 - m^2 r^2}}{mr} \right]. \quad (20)$$

Then $\Psi(T, r)$ provides the necessary change of coordinates from the soliton metric² ds_{sol}^2 in (17) to the black hole metric ds_{bh}^2 in (18).

It is useful to note that the inverse of the above map $\Psi^{-1} := \Theta = (\theta_1, \theta_2)$ isolates the dilaton τ . In fact,

$$T = \theta_1(x, t) = -\frac{1}{2mv} \log \left[\frac{a \tanh(\rho(x, t)) + 1}{a \tanh(\rho(x, t)) - 1} \right] + \frac{x}{v} \quad (21)$$

$$r = \theta_2(x, t) = \frac{a}{m} \operatorname{sech}(\rho(x, t)) = \frac{\tau(x, t)}{m} \quad (22)$$

□

²Observe that for any function ρ , we have the following trigonometric reductions $\cos(u/2) = -\tanh(\rho)$, $\sin(u/2) = \operatorname{sech}(\rho)$.

10 A Second Example

We can formulate an analogous result to Williams' Theorem using the oscillating kink-antikink solution of the sine-Gordon equation.

PROPOSITION 3. (*Beheshti*) Let $u = u(x, t) = 4 \arctan \left(\frac{v \sinh(amx)}{a \cos(vmt)} \right)$, $a^2 = 1 + v^2$ be the oscillating kink-antikink soliton solution of the sine-Gordon equation, from which we construct the soliton metric

$$ds^2 = \cos^2 \left(\frac{u}{2} \right) dx^2 - \sin^2 \left(\frac{u}{2} \right) dt^2. \quad (23)$$

For τ defined by $\tau(x, t) = \frac{4v^2 am \sin(vmt) \sinh(amx)}{a^2 \cos^2(vmt) + v^2 \sinh^2(amx)}$, the pair (g, τ) solve the dilaton equations given in (14).

Moreover, there exists a coordinate transformation $\Theta(x, t) = (\theta_1, \theta_2)$ taking the black hole metric in (9) having $J = 0$ and $\lambda = m^2$ given by

$$ds^2 = (M - m^2 r^2) dT^2 - \frac{1}{(M - m^2 r^2)} dr^2, \quad \text{for } M = 4A^2 v^4 m^2, \quad (24)$$

to the soliton metric in (23).

Proof. Let $\Theta(x, t) = (\theta_1(x, t), \theta_2(x, t)) = (T, r)$ be defined by

$$\theta_1(x, t) = \frac{x}{2Amv^2} + \frac{1}{2Am^2v^2} \operatorname{arctanh} \left(\frac{a^2 \cos^2(vmt) + (v^2 + 2) \sinh^2(amx)}{2a \cosh(amx) \sinh(amx)} \right) \quad (25)$$

$$\theta_2(x, t) = \frac{\tau(x, t)}{m}. \quad (26)$$

Then $\Theta(x, t)$ provides the necessary change of coordinates from the soliton metric ds_{sol}^2 in (23) to the black hole metric ds_{bh}^2 in (24). Note that in both cases, the black hole metric differs only by the constant M ; in this case, $M = 4A^2v^4m^2$, whereas in the Williams Theorem, $M = v^2$. \square

11 Restating the Eigenvalue Problem

Now that we have seen some concrete transformations, there are a few key observations we may make to generalise these results. First recall the gradient of a function $\tau(x_1, x_2)$ with respect to a diagonal metric g is given by the formula

$$\nabla\tau \stackrel{\text{def}}{=} \sum_{k=1}^2 g^{k1} \frac{\partial\tau}{\partial x_k} \frac{\partial}{\partial x_1} + \sum_{k=1}^2 g^{k2} \frac{\partial\tau}{\partial x_k} \frac{\partial}{\partial x_2},$$

so that $|\nabla\tau|_g^2 = g(\nabla\tau, \nabla\tau)$.

LEMMA 4. (*Conservation Law for τ*) Let $u = u(x, t)$ be a solution of the Euclidean sine-Gordon equation. Given a soliton metric g such that the pair (g, τ) solves the field equations (14), there exists a constant M such that

$$\frac{|\nabla\tau|_{sol}^2}{m^2} = \tau^2 - M.$$

In the case of Theorem (2), $M = v^2$; in the case of Proposition (3), $M = 4A^2v^4m^2$.

12 A Key Proposition

We also observe that in both results, the components of the transformations labeled $\Theta(x, t)$ satisfy the following proposition.

PROPOSITION 5. *Let $ds^2 \stackrel{g}{=} \cos^2(\frac{u}{2})dx^2 - \sin^2(\frac{u}{2})dt^2$ be a soliton metric so that (g, τ) solve the dilaton equations in (14). If $\Theta(x, t) = (\theta_1(x, t), \theta_2(x, t))$ is a coordinate transformation satisfying*

$$\frac{\partial \theta_1}{\partial x} = \frac{m \cot(\frac{u}{2}) \tau_t}{|\nabla \tau|_{sol}^2} \quad (27)$$

$$\frac{\partial \theta_1}{\partial t} = \frac{m \tan(\frac{u}{2}) \tau_x}{|\nabla \tau|_{sol}^2} \quad (28)$$

$$\theta_2 = \frac{\tau}{m}, \quad (29)$$

and $\Psi(T, r) = \Theta^{-1}(x, t)$, then $\Psi = \Psi(T, r)$ transforms the soliton metric to the black hole metric

$$\begin{aligned} ds_{bh}^2 &= \frac{-|\nabla \tau|_{sol}^2 \circ \Psi}{m^2} dT^2 + \frac{m^2}{|\nabla \tau|_{sol}^2 \circ \Psi} dr^2 \\ &= [M - (\tau \circ \Psi)^2] dT^2 - \frac{1}{[M - (\tau \circ \Psi)^2]} dr^2. \end{aligned} \quad (30)$$

As a consequence, we may easily compute the Laplacian of the above (very general!) black hole metric to be

$$\begin{aligned} \nabla_{bh}^2 &= \frac{2(\tau \circ \Psi) \frac{\partial(\tau \circ \Psi)}{\partial T}}{[M - (\tau \circ \Psi)^2]^2} \frac{\partial}{\partial T} + \frac{1}{[M - (\tau \circ \Psi)^2]} \frac{\partial^2}{\partial T^2} \\ &\quad + 2(\tau \circ \Psi) \frac{\partial(\tau \circ \Psi)}{\partial r} \frac{\partial}{\partial r} + [(\tau \circ \Psi)^2 - M] \frac{\partial^2}{\partial r^2}. \end{aligned} \quad (31)$$

Once we have established the form of the resulting black hole metric and its Laplacian, it is straightforward to translate our sine-Gordon eigenvalue problem to a black hole eigenvalue problem.

13 The Main Result

THEOREM 6. *Let g be a soliton metric with sine-Gordon soliton solution $u = u(x, t)$ such that (g, τ) solve the dilaton equations given in (7)=(14). Assume $\Theta(x, t)$, $\Psi(T, r) = \Theta^{-1}(x, t)$ are coordinate transformations satisfying the equations given in (5), so that the soliton metric ds_{sol}^2 in (8) is transformed to black hole metric ds_{bh}^2 in (30). Then for any function $f = f(T, r)$, one has*

$$(\nabla_{sol}^2(f \circ \Theta)) \circ \Psi = \nabla_{bh}^2 f, \quad (32)$$

That is, $\nabla_{sol}^2(f \circ \Theta) = (\nabla_{bh}^2 f) \circ \Theta$, or the Laplacians commute with the transformations.

As an immediate consequence, we connect sine-Gordon theory to black hole theory.

COROLLARY 7. *Let $\nu \in \mathbb{R}$. Under the same hypotheses as above, $f(T, r)$ satisfies the field equation/eigenvalue problem*

$$\nabla_{bh}^2 f = \nu f \quad (33)$$

if and only if $F(x, t) = f \circ \Theta(x, t)$ satisfies the field equation/eigenvalue problem

$$\nabla_{sol}^2 F = \nu F. \quad (34)$$

14 Hypergeometric function solutions

Finally, to solve the soliton eigenvalue problem, we need only solve the black hole eigenvalue problem. Under the hypotheses of Theorem (6), we may simplify the black hole Laplacian significantly

$$\nabla_{bh}^2 = \frac{1}{M - m^2 r^2} \frac{\partial^2}{\partial T^2} - (M - m^2 r^2) \frac{\partial^2}{\partial r^2} + 2m^2 r \frac{\partial}{\partial r}. \quad (35)$$

To solve $\nabla_{bh}^2 f = \nu f$, we first use a separation of variables $f(T, r) = e^{\omega T} h(r)$ to reduce the PDE to the ODE

$$B(r)^2 h''(r) - 2m^2 r B(r) h'(r) + (2m^2 B(r) - \omega^2) h(r) = 0, \quad (36)$$

where $B(r) \stackrel{def.}{=} M - m^2 r^2$, for $M = \text{constant}$.

Equation (36) has the form

$$\sigma(r)^2 u''(r) + \sigma(r) \tilde{\tau}(r) u'(r) + \tilde{\sigma}(r) u(r) = 0 \quad (37)$$

where $\sigma = B$, $\tilde{\tau} = -2m^2 r$, $\tilde{\sigma} = 2m^2 B - \omega^2$ are polynomials in r with $\deg \sigma$, $\deg \tilde{\sigma} \leq 2$, $\deg \tilde{\tau} \leq 1$. Such equations are classified as having *generalised hypergeometric type*, in the sense of A. Nikiforov and V. Uvarov [4]. Using the theory established in [1, 4, 7], we may construct explicit solutions of (33) and consequently solutions of (34). It is important to note that the techniques involved produce some integrality (or quantization) conditions.

15 Explicit solutions computed

EXAMPLE 3. (Kink soliton) Consider our first example having a soliton metric with $u(x, t) = 4 \arctan(e^{\pm\rho})$, where $\rho = \frac{m}{a}(x - vt)$, and $a^2 = 1 + v^2$. We have deduced that the resulting black hole metric satisfies $M = v^2$. Under the quantization conditions outlined in [7] and by superposition, one finds for $\nu = 2m^2$, the solutions

$$\begin{aligned} f_1(T, r) &= \sqrt{M - m^2 r^2} \sinh m\sqrt{MT} \\ f_2(T, r) &= \sqrt{M - m^2 r^2} \cosh m\sqrt{MT}. \end{aligned} \quad (38)$$

Thus, for Θ given in Theorem (2), we obtain solutions to the original eigenvalue problem $F_j = f_j \circ \Theta$, $j = 1, 2$

$$\begin{aligned} F_1(x, t) &= \frac{e^{mx}[a \tanh \rho - 1] - e^{-mx}[a \tanh \rho + 1]}{2} \\ F_2(x, t) &= \frac{e^{mx}[a \tanh \rho - 1] + e^{-mx}[a \tanh \rho + 1]}{2}. \end{aligned} \quad (39)$$

A third quantization condition will yield an infinite family of solutions, stated iteratively:

$$f_n^\pm(T, r) = e^{\pm m\sqrt{M(n+2)}T} (M - m^2 r^2)^{\frac{n+2}{2}} \frac{d^n}{dr^n} (M - m^2 r^2)^{-2} \quad (40)$$

yielding solutions $F_n^\pm = f_n^\pm \circ \Theta$; for details, consult [1].

16 A second explicit solution (preliminary findings)

EXAMPLE 4. (Oscillating kink-antikink) Consider our second example having a soliton metric with $u(x, t) = 4 \arctan\left(\frac{v \sinh(amx)}{a \cos(vmt)}\right)$, where $a^2 = 1 + v^2$. We have deduced that in this case the resulting black hole metric satisfies $M = 4A^2v^4m^2$. Since similar quantization conditions hold, we produce initial (T, r) solutions precisely as in the previous example for $\nu = 2m^2$

$$\begin{aligned} g_1(T, r) &= \sqrt{M - m^2r^2} \sinh m\sqrt{MT} \\ g_2(T, r) &= \sqrt{M - m^2r^2} \cosh m\sqrt{MT}. \end{aligned} \quad (41)$$

However, in this case, composition with the map Θ found in Theorem (3) produces different solutions to the original eigenvalue problem; $G_j = g_j \circ \Theta$, $j = 1, 2$ yields

$$\begin{aligned} G_1(x, t) &= 2Amv^2 \sqrt{1 - \frac{4a^2 \sin^2(vmt) \sinh^2(amx)}{(a^2 \cos^2(vmt) + v^2 \sinh^2(amx))^2}} \sinh(\alpha) \\ G_2(x, t) &= 2Amv^2 \sqrt{1 - \frac{4a^2 \sin^2(vmt) \sinh^2(amx)}{(a^2 \cos^2(vmt) + v^2 \sinh^2(amx))^2}} \cosh(\alpha), \end{aligned} \quad (42)$$

where we find α in terms of θ_1 computed in Theorem (3)

$$\alpha = \left(mx + \operatorname{arctanh} \left[\frac{a^2 \cos^2(vmt) + (v^2 + 2) \sinh^2(amx)}{2a \cosh(amx) \sinh(amx)} \right] \right). \quad (43)$$

Exploration concerning the possibility of using a third quantization condition or further analysis to produce an infinite family of solutions similar to the previous example is still in progress.

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