Wave Maps (and the Einstein Equations)

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Abstract

Wave Maps are harmonic maps from a Minkowski space into a Riemannian manifold. We will give a variational definition of the wave maps, and derive the wave maps system in covariant form, as well as in local coordinates on the target manifold. It will also be shown how the Wave Maps system arises in the reduction of the Einstein Vacuum Equations under special symmetry conditions, as well as the connection with the Yang-Mills system. Global existence, regularity and stability questions for the wave maps system, in both geometric and analytic contexts, will then be discussed. Some recent large data blow up results will also be mentioned.

1 Introduction

A map $\phi : M \to N$ between two Riemannian manifold (M^m, g) and (N^l, h) is called a **harmonic map** if it is the critical point of the Lagrangian

$$L_h(\phi) = \int_M g^{\alpha\beta} < \partial_\alpha \phi, \partial_\beta \phi >_h d\, vol_M$$

with respect to compactly supported variations. Here $g^{\alpha\beta}$ are the components of the metric g in local coordinates $x = (x^\alpha)_{1 \leq \alpha \leq m}$ on M, and and $\partial_\alpha = \partial_{x^\alpha}$.

Assume $N \subset \mathbb{R}^n$ isometrically for some n (by Nash's embedding theorem), then $\phi : M \to \mathbb{R}^n$ with $\phi \in N$, and the Euler-Lagrange equations for the above variational problem can be written as

$$\Delta_M \phi \perp T_\phi N \, (\subset T_\phi \mathbb{R}^n)$$

Where $\Delta_M = \frac{1}{\sqrt{|g|}} \partial_\alpha (g^{\alpha\beta} \sqrt{|g|} \partial_\beta)$ is the Laplace-Beltrami operator on M.

Equivalently using the second fundamental form $A^i_{jk}(\phi)$ of $N \subset \mathbb{R}^n$, the equations can be written as

$$\Delta_M \phi^i + A^i_{jk}(\phi)(\partial_\alpha \phi^j, \partial_\beta \phi^k)_{g^{\alpha\beta}} = 0$$

Take $M = \mathbb{R}^m$, and let D be the connection on the pull-back bundle $\phi^*(TN)$ induced by the Levi-Civita connection on the target manifold N, *i.e.* for V, a section of $\phi^*(TN)$ we set

$$D_X V = \nabla_{\phi^* X} V$$

and respectively $D_\alpha V = \nabla_{\partial_\alpha \phi} V$.

With this notation the Euler-Lagrange equations of the variational problem take the form

$$D_\alpha \partial_\alpha \phi = 0$$

Or utilizing the local coordinates on the target N, the equations can be written as

$$\Delta\phi^i + \Gamma^i_{jk}\partial_\alpha\phi^j\partial_\alpha\phi^k = 0$$

where Γ's are the Christoffel symbols on N.

This equation (system) is semilinear *elliptic* and is well understood from the analytic point of view.

If instead of a Riemannian target one takes the Minkowski space $M = \mathbb{R}\times\mathbb{R}^n$, with Lorentz metric $(g_{\alpha\beta}) = \mathrm{diag}(-1,1,...,1)$ as the domain, then the critical points of the Lagrangian

$$L_h(\phi) = \int_M g^{\alpha\beta} < \partial_\alpha\phi, \partial_\beta\phi >_h d\,vol_M$$

(under compactly supported variations) are called Minkowski harmonic maps, or **wave maps**. In this case the Euler-Lagrange equations take the form

$$D^\alpha\partial_\alpha\phi = 0$$

where the Lorentz metric $g^{\alpha\beta}$ is used to raise and lower the indices ($D^\alpha = g^{\alpha\beta}D_\beta$). In the local coordinates on the target manifold N, the equations can be written as

$$\Box\phi^i + \Gamma^i_{jk}\partial_\alpha\phi^j\partial^\alpha\phi^k = 0$$

Which is now a hyperbolic system of coupled semilinear equations.

As before one can use the embedding $N \subset \mathbb{R}^m$ for some m, to write the equations in the extrinsic form

$$\Box\phi \perp T_\phi N \qquad \text{or} \qquad \Box\phi + A^i_{jk}(\phi)(\partial_\alpha\phi^j, \partial^\alpha\phi^k) = 0$$

Where as before $A(\phi)$ is the second fundamental form of $N \subset \mathbb{R}^m$.

2 Connection with Einstein's equations and Yang-Mills fields

- Einstein's vaccum equations are given by

$$\mathbf{R}_{\alpha\beta}(g) = 0$$

where g is the Lorentzian metric of a 4 dimensional manifold and $\mathbf{R}_{\alpha\beta}$ is its Ricci tensor

$$\mathbf{R}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha \partial_\mu g_{\nu\beta} + \partial_\beta \partial_\mu g_{\nu\alpha} - \partial_\alpha \partial_\beta g_{\mu\nu} - \partial_\mu \partial_\nu g_{\alpha\beta})$$

In wave coordinates x^α, which are defined by

$$\Box_g x^\alpha = \frac{1}{|g|} \partial_\mu (g^{\mu\nu} |g| \partial_\nu) x^\alpha = 0$$

the Einstein vacuum equations take the reduced form

$$g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} = N_{\mu,\nu}(g, \partial g)$$

where $N(g, \partial g)$ is quadratic in the first order derivatives ∂g. Thus in this reduced form the equations have the form of quasi-linear hyperbolic equations. Under $U(1)$ symmetry reduction the reduced system can be expressed as a wave map system for $\phi : M^{1+2} \to \mathbb{H}^2$. Cf. Klainerman (1997) [5], Moncrief (1988)[15], Misner (1978)[14].

- Suppose $\phi : R^{1+n} \to N$ is a wave map, and the target manifold N is rotationally symmetric with the metric given by

$$ds^2 = du^2 + g^2(u) d\chi^2$$

where $d\chi^2$ is the standard metric on the sphere $\mathbb{S}^{k-1} \subset \mathbb{R}^k$. ($N$ can be identified with a ball of radius $R \in \mathbb{R}^+ \cup \{\infty\}$ in $\mathbb{R}^k$)

Then in the spatial polar coordinates $(t, \rho, \omega) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{n-1}$ on the domain, the wave map system takes the form ($\phi = (u, \chi)$)

$$u_{tt} - u_{\rho\rho} - \frac{n-1}{\rho} u_\rho + \frac{k}{\rho^2} g(u) g'(u) = 0$$

which is a scalar wave equation for the spatially radial function $u : \mathbb{R}^{1+n} \longrightarrow \mathbb{R}$.

Let P be a principal fiber bundle over the manifold M with structure group G, and canonical projection π. $\mathfrak{g}$ is the Lie algebra of G. A connection A can be thought of locally as a $\mathfrak{g}$-valued 1-form $A = A_\mu(x) dx^\mu$. The curvature of the connection A is defined to be the 2-form $F = F_{\mu\nu} dx^\mu dx^\nu$ with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Then the Yang-Mills Lagrangian is

$$L(A) = \int_M F_{\mu\nu} F^{\mu\nu} d\, vol_M$$

and the Euler-Lagrange equations have the form

$$D^\mu F_{\mu\nu} = 0$$

For the particular case $M = \mathbb{R}^{1+d}$ the Minkowski space, and $G = SO(d)$, the group of orthogonal transformations on $\mathbb{R}^d$ we have that $A_\mu(x)$ is a $d \times d$ skew-symmetric matrix A_μ^{ij}. The appropriate equivariant ansatz has the form

$$A_\mu^{ij}(x) = (\delta_\mu^i x^j - \delta_\mu^j x^i) h(t, |x|)$$

where $h : M \to \mathbb{R}$ is a spatially radial function. Setting $u = r^2 h$ where $r = |x|$, and $n = d - 2$, the Yang-Mills system reduces to the following scalar equation for u

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r + \frac{2n}{r^2} u(1-u) \left(1 - \frac{1}{2} u\right) = 0$$

3 Local theory, Scaling

We study the following Cauchy problem for the map $\phi : \mathbb{R} \times \mathbb{R}^n \to N$

$$\Box\phi^\alpha + \Gamma^\alpha_{jk}\partial^i\phi^j\partial_i\psi^k = 0$$
$$\phi(0,\cdot) = f, \phi_t(0,\cdot) = f$$

Where the initial data (f, g) will be generally taken to live in a suitable Sobolev space.

Schematically the equations have the form

$$\Box\phi = N(\phi, \nabla\phi)$$

Which we heuristically rewrite as

$$\phi = S(f, g) + \Box^{-1}N(\phi)$$

with $S(f, g)$ being the solution operator for the free wave equation.

The question of the *Local Well Posedness, LWP*, (existence, continuous dependence on initial data, persistence of higher regularity) for the initial data $(f, g) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ reduces to finding a fixed point for small time in Banach space X, Y, for which

$$S : H^s \times H^{s-1} \to X, \quad \Box^{-1} : Y \to X, \quad N : X \to Y.$$

For $s > \frac{n}{2} + 1$ this can be achieved by energy methods, taking

$$X = \{\phi \,|\, \phi \in L^\infty(H^s), d\phi \in L^\infty(H^{s-1})\}, \quad Y = L^1(H^{s-1}).$$

The wave map system is invariant under the scaling

$$\phi_\lambda(t, x) = \phi(\lambda t, \lambda x)$$

The $\dot{H}^s$ norm of the scaled initial data will scale as $\|\phi_\lambda(0,\cdot)\|_{\dot{H}^s} = \lambda^{s-\frac{n}{2}}\|\phi(0,\cdot)\|_{\dot{H}^s}$, thus the Sobolev norm with exponent $s_c = \frac{n}{2}$ is invariant under scaling. There are the following equivalence between LWP and GWP for different exponents.

- $s < s_c$ 'small data, small time' is equivalent to 'large data, large time'. Expect ill-posedness.
- $s = s_c$ 'small data, small time' is equivalent to 'small data, large time'. Provided the existence time depends only on the size of the norm, same holds for large data. Expect global regularity.
- $s > s_c$ 'small data, large time' is equivalent to 'large data, small time'

Notice that the energy methods allow one to obtain LWP only for $s > s_c+1$

The special structure of the nonlinearity in our problem, which can be written as $N(\phi) = \Gamma(u)Q_0(\phi, \psi)$, where

$$Q_0(u, w) = \partial_\alpha u \partial_\alpha v = u_t v_t - \nabla u \cdot \nabla v$$

allows one to establish LWP up to the critical scaling, which was used by Klainerman & Machedon ($n \geq 3$)[6], Klainerman & Selberg ($n = 2$)[8], who utilized the spaces

$$(X, Y) = X^{s,b} \times X^{s-1,b-1+}$$

to establish LWP in H^s for $\forall s > s_0$. The Wave-Sobolev space $X^{s,b}$ is defined as

$$X^{s,b} = \{\phi \, : \, \|(1 + |\xi|^2)^{\frac{s}{2}}(1 + |\tau^2 - |\xi|^2|)^{\frac{b}{2}}\hat{\phi}\|_{L^2} < \infty\}.$$

Compare to

$$H^s = \{\phi \, : \, \|(1 + |\xi|^2)^{\frac{s}{2}}\hat{\phi}\|_{L^2} < \infty\}.$$

4 Global Theory

- If $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a smooth solution to the linear scalar wave equation $\Box u = 0$, and $\gamma : \mathbb{R} \to N$ is a geodesic into the target manifold, then the map $\phi(t,x) = \gamma(u(t,x))$ from $\mathbb{R} \times \mathbb{R}^n \to N$ is a global smooth wave map.

$$D^\alpha \partial_\alpha \gamma(u) = \Box u \dot{\gamma}(u) + \partial_\alpha u D^\alpha \dot{\gamma}(u) = 0$$

- In 1+2 dimension global regularity for spherically symmetric wave maps $\phi(t,x) = \phi(t,|x|)$ was obtained by Christodolou & Tahvildar-Zadeh, (1993) [2] for geodesically convex targets N, which was later generalized by Struwe to include the sphere (2000) [24] and later to any Riemannian manifold without boundary (2001) [25].

- In the case of rotationally symmetric targets N similar to the previous results were obtained for co-rotational (equivariant) wave maps into geodesically convex targets by Shatah & Tahvildar-Zadeh (1992)[20], which was later generalized to more general targets by Grillakis (1992) [4], and Struwe (2003) [26]. The numerical work of Bizon et al. (2000)[1] suggested blow up in the case $N = S^2$

- Under smallness assumption on the data one can obtain GWP in the Besov space $\dot{B}^{\frac{n}{2},1} \times \dot{B}^{\frac{n}{2}-1,1}$. This was established by Tataru first for $n \geq 4$ (1998)[29], then in the remaining cases $n = 2, 3$ (1998) [30]. More specifically Tataru showed that the solution to the Caucy problem

$$\Box \psi^\alpha + \Gamma^\alpha_{jk} \partial^i \psi^j \partial_i \psi^k = 0$$
$$\phi(0,\cdot) = f, \phi_t(0,\cdot) = g$$

with initial data satisfying

$$\|(f,g)\|_{\dot{B}^{2,1}_{\frac{n}{2}} \times \dot{B}^{2,1}_{\frac{n}{2}-1}} < \epsilon$$

has global solution, which is a unique limit of smooth solutions, and the solution map depends Lipschitz on the initial data.

Under the assumption of smallness of initial data in the Besov space, Tataru can work in local coordinates because of the embedding $\dot{B}^{\frac{n}{2},1} \hookrightarrow L^\infty$, which prevents the solution from exiting the chart domain. This essentially allowes one to completely forget about the geometry of the target manifold.

The spaces $\dot{B}^{s,p}$ can be defined by decomposing the functions in Fourier space and using Littlewood-Paley theory. If $f = \sum_{k\in\mathbb{Z}} P_k f$ is the L-P decomposition, then for $1 \le p < \infty$ we have

$$\|f\|_{\dot{B}^{s,p}} = \left(\sum_{k\in\mathbb{Z}} (2^{sk}\|P_k f\|_{L^2})^p\right)^{\frac{1}{p}} = \left\|\{2^{sk}\|P_k f\|_{L^2}\}_{k\in\mathbb{Z}}\right\|_{l^p}$$

Thus

$$\|f\|_{\dot{B}^{\frac{n}{2},1}} = \left\|\{2^{\frac{nk}{2}}\|P_k f\|_{L^2}\}_{k\in\mathbb{Z}}\right\|_{l^1} \quad \text{vs.} \quad \|f\|_{\dot{H}^{\frac{n}{2}}} = \left\|\{2^{\frac{nk}{2}}\|P_k f\|_{L^2}\}_{k\in\mathbb{Z}}\right\|_{l^2}$$

The methods used in dimensions $n \ge 4$, and $n = 2, 3$ are different, due to the lack of appropriate Strichartz estimates in lower dimensions. The later is compensated by the use of 'null-frame spaces' $L^p_{t_\omega}(L^q_{x_\omega})$, in which a variant of $L^2(L^\infty)$ Strichartz estimate holds, allowing one to close the estimates in the space $\dot{X}^{s,b,1}$.

- Global regularity for general small data in the critical Sobolev space $\dot{H}^{\frac{n}{2}}$ was first obtained by Tao (2000) for $n \ge 5$ [27] using only Strichartz estimates, while managing to take care of logarithmic divergences that arise in the l^2 case. In this case one can't simply use the formulation in local coordinates, though Tao considers only the case of $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ targets, and uses the extrinsic formulation to write the equations as

$$\Box\phi = -\phi(\partial_\alpha\phi \cdot \partial^\alpha\phi)$$

After decomposing in frequency $\phi = \sum_{k\in\mathbb{Z}} P_k\phi$ he manages to take care of all the interactions in the nonlinearity with the exception of $\phi_{low}\partial_\alpha\phi_{low}\partial^\alpha\phi_{high}$, which is then 'gauged away' by a microlocal gauge.

To see the significance of the gauge in the wave maps system consider an orthonormal frame $e_1, e_2, ..., e_k$ in TN. Given a smooth map $\phi : \mathbb{R}^{1+n} \to N$ one can pull-back this frame to the pull-back bundle $\phi^*(TN)$. Then the derivatives of the map ϕ can be written in this frame as

$$\partial_\alpha\phi = e\psi_\alpha$$

and the covariant derivative has the form

$$D_\alpha = \partial_\alpha + A_\alpha$$

where $A = A_\alpha dx^\alpha$ is a matrix valued 1-form.

Using the wave map equations, the definition of the connection form A, and the zero torsion identities

$$D_\alpha\psi_\beta = D_\beta\psi_\alpha,$$

we can write the derivative formulation of the wave map system

$$\begin{aligned}
&\partial_\alpha\phi = e\psi_\alpha\\
&(\phi^*\nabla)_\alpha = eA_\alpha\\
&D^\alpha\psi_\alpha = 0\\
&D_\alpha\psi_\beta - D_\beta\psi_\alpha = 0\\
&F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]
\end{aligned}$$

This system is undertermined, since one has freedom of choice of the frame e.

- The above derivative formulation was used by Klainerman & Rodnianski (2001)[7] to extend Tao's result to general targets, as well as simultaneously by Shatah & Struwe (2002) [22], and Nahmod-Stefanov-Uhlenbeck

(2002)[16] to give alternative proofs in dimensions $n \geq 4$. The Shatah-Struwe result is particularly interesting, since it simplifies the proof significantly by essentially only using Strichartz estimates and no microlocalization. In the last two papers the Coulomb gauge $\sum \partial_i A_i$ was used, which gives elliptic equations for the connection form A

$$\Delta A_\beta + \partial_i [A_i, A_\beta] = \partial_i F_{i\beta} = \partial_i (R(\partial_i \phi, \partial_\beta \phi), \quad 0 \leq \beta \leq k$$

- For the low dimensional cases $n = 2, 3$ the global regularity in the critical Sobolev norm was again obtained by Tao (2000) [28], where he again looks at only $\mathbb{S}^k$ targets and analyzes the equation

$$\Box \phi = -\phi(\partial_\alpha \phi \cdot \partial^\alpha \phi)$$

Using microlocalization and Tataru's null-frame spaces to compensate for the missing Strichartz's estimates, Tao again manages to control all the interactions except one, which is again gauged away by a microlocal gauge. The paper is extremely technical, and the solution space looks like ($\sim 2^k$ frequency terms)

$$\begin{aligned} \|\phi\|_{S[k]} = \|\nabla_{x,t}\phi\|_{L_t^\infty \dot{H}_x^{n/2-1}} + \|\nabla x, t\phi\|_{\dot{X}_k^{n/2-1,1/2,\infty}} \\ + \sup_{\pm} \sup_{l>10} (\sum_{\kappa \in K_l} \|P_{k,\pm\kappa} Q^{\pm}_{<k-2l}\phi\|^2_{S[k,\kappa]})^{1/2} \end{aligned}$$

where

$$\|\phi\|_{S[k,\kappa]} = 2^{nk/2}\|\phi\|_{NFA^*[\kappa]} + |\kappa|^{-1/2} 2^{k/2}\|\phi\|_{PW[\kappa]} + 2^{nk/2}\|\phi\|_{L_t^\infty L_x^2}$$

- Krieger extended Tao's low dimensional result to the hyperbolic target $\mathbb{H}^2$ first for $n = 3$ (2003)[10], then for $n = 2$ (2004)[11]. He used essentially the same spaces, while utilizing the Coulomb gauge at the beginning similar to Shatah-Struwe and Nahmod-Stefanov-Uhlenbeck (rather than after microlocalization as was done by Tao and Klainerman-Rodnianski).

- For general targets that can be 'uniformly isometrically embedded' into some euclidean space $\mathbb{R}^k$, the global regularity for small data in the critical Sobolev space was obtained by Tataru (2004) [31] by using similar tools.

5 Stability, Blow up

- A variant of orbital stability for the geodesic wave map $\phi = \gamma(u)$ in R^{1+3} was established by Sideris (1989) [23], where a global smooth solution to the wave maps system is constructed by a perturbation of the geodesic wave map. This perturbed solution remains in a tubular neighborhood around the geodesic for all time. The spaces used are essentially $X^{s,b}$ type with $s > 10$, and the tools are just energy estimates based on embeddings of $X^{s,b}$ spaces, and bilinear null-form estimates in these spaces established by Klainerman in a series of earlier papers. Due to the high regularity, one essentially doesn't see the global geometry of the target manifold.

- Recently Krieger (2005) [12] obtained stability of spherically symmetric, and geodesic wave maps $\phi : \mathbb{R}^{1+2} \to \mathbb{H}^2$. The stability of spherical maps is based on the asymptotic behavior of such maps established by Christodolou & Tahvildar-Zadeh (1993) [3]. The stability is in the sense of the closeness of the (Coulomb) gauged derivative components of the perturbed map to the spherical symmetric one in the L^2 sense. Thus it's not clear from the result whether the map itself (not the gauged components) remains close to the spherical symmetric map.

- It's possible to obtain stability of the geodesic wave map by using the exponential map on the target manifold N to compare the 'difference' between the perturbed map, and the geodesic one. In the Fermi chart in a neighborhood of the geodesic γ, the 'difference equations' look similar to the wave map system in local coordinates. To illustrate this we concentrate on the case $\phi : \mathbb{R}^{1+n} \to \mathbb{S}^2$

  The geodesic map is denoted by $\varphi(t,x) := \gamma(u(t,x))$, where $\gamma : \mathbb{R} \to \mathbb{S}^2$ is a geodesic and $u : \mathbb{R}^{1+n} \to \mathbb{R}$ is a free wave, *i.e.* $\Box u = 0$.

  For notation convenience define $\rho(t,x) := \gamma(u(t,x) + s(t,x))$

  The perturbed map is then $\psi = \exp_\rho \vec{v}$, where $\vec{v} = v\vec{n}$.

Differentiating the expression of ψ in the direction of x_α ($\alpha = 0, 1, \ldots, n$) we get:

$$\partial_\alpha \psi = \partial_\alpha v \vec{n}_\psi + \cos v (\partial_\alpha u + \partial_\alpha s) \vec{t}_\psi$$

(Consider $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and the family of geodesics $\Psi(v, r) = (\cos v \cos r, \cos v \sin r, \sin v)$, then $|\partial_t \Psi(v, r)|^2 = \cos^2 v$).

Thus, the WM system for ψ will look like

$$\begin{aligned} 0 =& D^\alpha \partial_\alpha \psi = \\ & \Box v \vec{n} + \cos v Q_0(v, u+s) \nabla_{\vec{t}} \vec{n} + (\cos v \Box s + Q_0(\cos v, u+s)) \vec{t} \\ & + \cos v Q_0(v, u+s) \nabla_{\vec{n}} \vec{t} + \cos v Q_0(u+s, u+s) \nabla_{\vec{t}} \vec{t} \end{aligned}$$

And the $s - v$ system takes the form:

$$\begin{cases} \Box v + \cos v \left[(\Gamma^1_{12}(\psi) + \Gamma^1_{21}(\psi)) Q_0(v, u+s) + \Gamma^1_{22}(\psi) Q_0(u+s, u+s) \right] = 0 \\ \cos v \Box s + \cos v \left[(\Gamma^2_{12}(\psi) + \Gamma^2_{21}(\psi)) Q_0(v, u+s) + \Gamma^2_{22}(\psi) Q_0(u+s, u+s) \right] \\ \qquad + Q_0(\cos v, u+s) = 0 \\ \Box u = 0 \end{cases}$$

or simplifying (v is small, so $\cos v \neq 0$):

$$\begin{cases} \Box v + \cos v \left[(\Gamma^1_{12}(\psi) + \Gamma^1_{21}(\psi)) Q_0(v, u+s) + \Gamma^1_{22}(\psi) Q_0(u+s, u+s) \right] = 0 \\ \Box s + (\Gamma^2_{12}(\psi) + \Gamma^2_{21}(\psi)) Q_0(v, u+s) + \Gamma^2_{22}(\psi) Q_0(u+s, u+s) \\ \qquad + \tan v Q_0(v, u+s) = 0 \\ \Box u = 0 \end{cases}$$

Thus schematically the equations can be thought of as $\Box V = \Gamma(V) Q_0(V, V)$.

In this case similar to Tataru's result for GWP in Besov spaces, one obtains global existence for the perturbed map if initially $\|(v, s)\|_{\dot{B}^{n/2,1}}$ is small, which then stays close to the geodesic wave map for all time. Due to the embedding $\dot{B}^{n/2,1} \hookrightarrow L^\infty$, one has (strong) pointwise asymptotic stability.

- Finally I want to mention two very recent blow-up results for the large data wave maps $\phi : \mathbb{R}^{1+2} \to \mathbb{S}^2$, which are in sync with the numerical evidence obtained by Bizón et al. (2001)[1]. The first result is due to Rodnianski & Sterbenz (April 2006) [17] in which they constructively find a set of initial data that leads to development of singularities in finite time. This is done by considering a k-equivariant wave map as a perturbation of a self-similar time scaled harmonic map for $k \geq 4$.

  The second result is due to Krieger, Schlag and Tataru (October 2006) [13], in which they construct a 1-equivariant map as a perturbation of the time-scaled harmonic map. In this last result the blow up rate can be controled and made arbitrarily slow, while the initial data leading to this blow up is not generic.

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