A Non-linear Schrödinger Type Formulation of FLRW Scalar Field Cosmology

Jennie D'Ambroise
Department of Mathematics and Statistics
University of Massachusetts

Abstract

In this talk I will present a non-linear Schrödinger type formulation of Einstein's equations for a Friedmann-Lemaître-Robertson-Walker universe with scalar field and perfect fluid matter source. This provides for an alternate method of obtaining exact solutions of the Einstein field equations for a homogeneous, isotropic universe. Some examples of exact solutions obtained in this way will be demonstrated, and I will mention analogous work that is currently being done with a Bianchi metric in an anisotropic universe.

Einstein Equations

For coordinates $\{x_i\}_{i=1}^n$ and metric $g = \sum_{ij} g_{ij} dx_i dx_j$:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^n g^{lk}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right) \text{ are Christoffell Symbols,}$$

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x_i} - \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_{s=1}^n \left(\Gamma_{is}^l \Gamma_{jk}^s - \Gamma_{js}^l \Gamma_{ik}^s\right) \text{ is the Curvature Tensor,}$$

$$R_{ij} = \sum_l R_{ilj}^l \text{ is the Ricci Tensor,}$$

where g^{ij} denotes elements of the inverse matrix.
We will use units such that the speed of light is one.

We consider Einstein's Equations with zero cosmological constant:

$$R_{ij} - \frac{1}{2} g_{ij} R = -K^2 T_{ij}.$$

Einstein postulated that the energy-momentum tensor T_{ij} should be modeled on the stress tensor of a perfect fluid with density $\rho_T(t)$ and pressure $p_T(t)$. That is, he postulated that the energy-momentum tensor takes the form

$$T^{ij} = (\rho_T + p_T)\delta_{i1}\delta_{j1} + p_T g^{ij}.$$

FLRW field equations

For a Friedmann-Lemaître-Robertson-Walker (FLRW) universe, which is a homogenous, isotropic model with perfect fluid matter source, scalar field ϕ and potential V, density ρ, pressure p and the metric g take the form

$$g = ds^2 = -dt^2 + \frac{a^2}{1-kr^2}dr^2 + r^2a^2d\theta^2 + a^2r^2\sin^2\theta d\phi^2,$$

$$\rho_T = \frac{\dot{\phi}^2}{2} + V\circ\phi + \frac{D(t)}{a^n}$$

$$p_T = \frac{\dot{\phi}^2}{2} - V\circ\phi + \frac{(n-3)D(t)}{3a^n},$$

for coordinates $x_1 = t$, $x_2 = r$, $x_3 = \theta$ and $x_4 = \phi$,
curvature parameter $k = -1, 0$ or 1, and
scale factor $a = a(t)$ (the expansion rate of the universe).
Also, $n \neq 0$ is a constant and
$D(t)$ is a non-constant function of t.

Therefore by direct computation, Einstein's field equations in a FLRW universe take the form

$$H^2 + \frac{k}{a^2} \overset{(i)}{=} \frac{K^2}{3}[\frac{\dot{\phi}^2}{2} + V\circ\phi + \frac{D(t)}{a^n}]$$

$$\dot{H} + H^2 \overset{(ii)}{=} -\frac{K^2}{6}[-\dot{\phi}^2 + V\circ\phi - \frac{D(t)(n-2)}{2a^n}]$$

for $K^2 := 8\pi G$, G is Newton's constant, and $H := \frac{\dot{a}}{a}$ is the Hubble parameter.

Correspondence between FLRW field equations and a Schrödinger-type equation

In 2003, Floyd Williams and Panos Kevrekidis described a correspondence between the FLRW field equations and an Ermakov-Milne-Pinney (EMP) equation [5]. It is known that there is a further correspondence between the EMP equation and a Schrödinger-type equation, which was the motivation for the following theorem. These correspondences were the motivation for the following, which relates FLRW field equations directly with a non-linear Schrödinger-type equation.

Theorem 1. *Given n and k as above, $E(x)$ and $P(x)$, let $u(x)$ be a solution to the non-linear Schrödinger-type equation*

$$u'' + [E(x) - P(x)]u = -\frac{nk}{2}u^{(4-n)/n} \text{ (NLS)} .$$

Define $\sigma(t)$, $\psi(x)$ and $D(t)$ to be such that

$$\dot{\sigma}(t) = u(\sigma(t)), \quad \psi'(x)^2 = \frac{4}{nK^2}P(x) \quad \text{and} \quad D(t) = -\frac{12}{n^2K^2}E(\sigma(t))$$

Then the following triple solves the FRLW field equations:

$$a(t) = u(\sigma(t))^{-2/n} \quad \phi(t) = \psi(\sigma(t))$$

$$V = \left[\frac{12}{K^2n^2}(u')^2 + \frac{2u^2}{K^2n}\left(\frac{6}{n}E - P\right) + \frac{3k}{K^2}u^{\frac{4}{n}}\right] \circ \psi^{-1}.$$

Conversely,
given $D(t)$, n and k let (a, ϕ, V) be a solution of the FRLW field equations.
Define $\sigma(t)$, $\psi(x)$, $E(x)$ and $P(x)$ to be such that

$$\dot{\sigma}(t) = a(t)^{-n/2} \quad \psi(x) = \phi(\sigma^{-1}(x))$$

$$E(x) := -\frac{K^2 n^2}{12} D(\sigma^{-1}(x)) \quad P(x) = \frac{nK^2}{4} \psi'(x)^2.$$

Then a solution to NLS is

$$u(x) := a(\sigma^{-1}(x))^{-\frac{n}{2}}.$$

Examples from various papers in the literature have been reconstructed using this theorem.

This theorem can also be used to show that for $a(t)$ given, ϕ and V that satisfy the FRLW field equations take the form

$$\begin{aligned} \dot{\phi}(t)^2 &= -\frac{2}{K^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD(t)}{3a^n}, \\ V(\phi(t)) &= \frac{3}{K^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \frac{(n-6)D(t)}{6a^n} \end{aligned}$$

These equations are well-known in the literature.

Alternative Perspective If the scale factor $a(t)$, $D(t)$, k and n are first specified, then

$$\begin{aligned} a(t) \;&\Rightarrow\; u(x), E(x) \text{ via Theorem 1 (converse)} \\ &\Rightarrow\; \text{Solve for the remaining } P(x) \text{ using NLS equation} \\ &\Rightarrow\; \phi(t), V \text{ via Theorem 1.} \end{aligned}$$

Alternative Perspective with D(t)=0 In this case, n is not specified by the FRLW field equations

$\Rightarrow$ can choose n such that resulting u-data is computable

Since NLS is linear when $n = 2$ and $n = 4$, these are often convenient choices.

Example (reproduce example from literature) This example appears in a paper by Ellis and Madsen[3]. Let $k = 1$, $D = 0$ and

$$a(t) = A\cosh(\omega t),\ A, \omega > 0.$$

By the converse of Theorem 1, let $\sigma(t)$ be such that $\dot{\sigma} = a(t)^{-n/2}$. Here, $n = 4$ is a nice choice since then

$$\dot{\sigma} = \frac{1}{A^2}\text{sech}^2\omega t \quad \text{and} \quad \sigma(t) = \frac{1}{\omega A^2}\tanh\omega t$$

Still using Theorem 1 converse, $E = 0$ and

$$u(x) = a\left(\sigma^{-1}(x)\right)^{-n/2} = \frac{1}{A^2}\left(1 - \omega^2 A^4 x^2\right).$$

The function $P(x)$ is still unknown, so solving NLS for it, we find

$$P(x) = \frac{2A^2(1-\omega^2 A^2)}{1-\omega^2 A^4 x^2}.$$

Now that we have u-data, we may use the theorem (first implication) to get the ϕ and V which satisfy the FRLW field equations. According to the theorem, we need ψ such that

$$\psi'(x)^2 = \frac{1}{K^2}P(x) = \frac{2A^2(1-\omega^2 A^2)}{K^2(1-\omega^2 A^4 x^2)}$$

Therefore for $A^2\omega^2 < 1$, $\psi(x) = \frac{\sqrt{2(1-\omega^2 A^2)}}{\omega K A}\text{Arcsin}(\omega A^2 x) + \psi_o$ for ψ_o an integration constant. Finally, ϕ and V take the form

$$\phi(t) = \psi(\sigma(t)) = \frac{\sqrt{2(1-\omega^2 A^2)}}{\omega K A}\text{Arcsin}(\tanh\omega t) + \psi_o$$

$$V(y) = \frac{2(1-\omega^2 A^2)}{K^2 A^2}\cos^2\left(\frac{\omega K A}{\sqrt{2(1-\omega^2 A^2)}}(y - \psi_o)\right) + \frac{3\omega^2}{K^2}$$

Example (new) For $k = -8\lambda^2$, $n = 1$,

$$E(x) = x^2 + \frac{27\lambda^2}{4} \text{ and } P(x) = x^2,$$

a solution of NLS is

$$u(x) = -\frac{3\sqrt{3}}{4}\tanh(\sqrt{\frac{27}{8}}\lambda x).$$

Then according to the theorem, the functions below are constructed, some with the aid of Mathematica.

$$\sigma(t) = \frac{1}{\lambda}\sqrt{\frac{8}{27}}\text{Arcsinh}(e^{-\frac{27\lambda}{8\sqrt{2}}(t-c_1)})$$

$$\psi(x) = \frac{1}{K}x^2 + c_2$$

$$D(t) = -\frac{12}{K^2}\left(\frac{8}{27\lambda^2}\text{Arcsinh}^2\left(e^{-\frac{27\lambda}{8\sqrt{2}}(t-c_1)}\right) + \frac{27}{4}\lambda^2\right),$$

where c_1 and c_2 are integration constants. And therefore the following triple solves the FRLW field equations:

$$a(t) = \frac{16}{27}\left(1 + e^{\frac{27\lambda}{4\sqrt{2}}(t-c_1)}\right)$$

$$\phi(t) = \frac{8}{27K\lambda^2}\text{Arcsinh}^2\left(e^{-\frac{27\lambda}{8\sqrt{2}}(t-c_1)}\right) + c_2$$

$$V(y) = \frac{2187\lambda^2}{32K^2} + \frac{135}{8K}(y - c_2)\tanh^2\sqrt{\frac{27}{8}K\lambda^2(y - c_2)},\ y > c_2.$$

This example was inspired by a separate example in a paper by Dionisio Bazeia[1].

General Solution of Special Case Using Duffing Equation
By choosing

$$n = 1 \quad k = 2B < 0 \quad E(x) - P(x) = A > 0,$$

NLS reduces to the Duffing equation

$$u'' + Au = -Bu^3,$$

which has a general solution

$$u(x) = \sqrt{\frac{A}{-B}} \tanh\left(\sqrt{\frac{A}{2}}(x - c_1)\right) \text{ for some constant } c_1.$$

Then by the above theorem,

$$\sigma(t) = \sqrt{\frac{2}{A}} \text{Arcsinh}\left(e^{\frac{A}{\sqrt{-2B}}(t-c_2)}\right) + c_1 \text{ for some constant } c_2$$

$$\psi(t) = \pm\frac{2}{K}\left[\int \sqrt{P(x)}dx\right] \qquad D(t) = -\frac{12}{K^2}E(\sigma(t))$$

And the solution to the FLRW field equations is

$$a(t) = -\frac{B}{A}\left(e^{-\sqrt{\frac{2}{-B}}A(t-c_2)} + 1\right)$$

$$\phi(t) = \psi(\sigma(t))$$

$$V(\phi(t)) = -\frac{2A}{K^2B}\left(3A + \frac{5P(\sigma(t))}{\left(1 + e^{-\sqrt{\frac{2}{-B}}A(t-c_2)}\right)}\right)$$

Anisotropic universe with Bianchi I metric.

For a given scalar field $\phi(t)$, potential V, energy-momentum tensor

$$T_{ij} = -\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_j} + \frac{1}{2} g_{ij}\left[\sum_{k,s} g^{ks}\frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_s} + 2V \circ \phi\right],$$

and Bianchi I metric

$$g = -dt^2 + X(t)^2 dx^2 + Y(t)^2\left(dy^2 + dz^2\right),$$

the Einstein field equations take the form

$$2\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}^2}{Y^2} \overset{(a)}{=} K^2\left[\frac{1}{2}\dot{\phi}^2 + V \circ \phi\right]$$

$$2\frac{\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} \overset{(b)}{=} -K^2\left[\frac{1}{2}\dot{\phi}^2 - V \circ \phi\right]$$

$$\frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} + \frac{\ddot{X}}{X} \overset{(c)}{=} -K^2\left[\frac{1}{2}\dot{\phi}^2 - V \circ \phi\right].$$

A number of other equations can be established

- Conservation equation for $\rho = \frac{1}{2}\dot{\phi}^2 + V\circ\phi$ and $p = \frac{1}{2}\dot{\phi}^2 - V\circ\phi$

$$\dot{\rho} + 3H(\rho + p) = 0$$

- For $H := \frac{1}{3}\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)$ and $\mu := \frac{1}{\sqrt{3}}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)$, (a) can be rewritten as

$$3H^2 - \mu^2 \overset{(a)'}{=} K^2\left[\frac{1}{2}\dot{\phi}^2 + V\circ\phi\right]$$

- By direct computation and (a)-(c),

$$\dot{\mu} + 3H\mu = 0$$

- Using the above equations we can verify Raychaudhuri's equation

$$3\dot{H} + 3H^2 + 2\mu^2 \overset{(d)}{=} -\frac{K^2}{2}(\rho + 3p)$$

Now, the equations (a)' and (d) are exactly the field equations for an FRLW universe by choosing

$$n = 6 \quad k = 0 \quad a = \left[XY^2\right]^{1/3} \quad D(t) = \frac{1}{K^2}\mu^2 a^6$$

The goal is to obtain, for a given ϕ and V, the implication

$$
\begin{aligned}
& u(x) \quad \text{solving NLS for } E < 0 \text{ constant} \\
\Rightarrow \quad & a(t) \text{ solving FRLW field equations} \\
\stackrel{?}{\Rightarrow} \quad & (X, Y) \text{ solving Bianchi I field equations}
\end{aligned}
$$

? is possible by rewriting

$$X(t) = R(t)e^{\alpha(t)} \quad Y(t) = R(t)e^{\beta(t)} \text{ for some } \alpha + 2\beta = 0.$$

Then by examining the converse implication, $(X, Y) \Rightarrow (a, \phi, V) \Rightarrow u(x)$, we can make an ansatz for α:

$$\alpha = \frac{2}{3}\sqrt{-E}\sigma(t), \text{ for } \sigma \text{ such that } \dot{\sigma} = u \circ \sigma,$$

and R is given by

$$a = \left[XY^2\right]^{1/3} = \left[Re^{\alpha}R^2e^{2\beta}\right]^{1/3} = R,$$

thereby combining to give X and Y.

That is,

Theorem 2. *Given* $E(x) = E$ *constant and* $P(x)$*, let* $u(x)$ *be a solution to the linear Schrödinger-type equation*

$$u'' + [E(x) - P(x)]u = 0.$$

Define $\sigma(t)$ *to be such that*

$$\dot{\sigma(t)} = u(\sigma(t)).$$

Then for $a(t) = u(\sigma(t))^{-1/3}$ *and* $\alpha(t) = \frac{2}{3}\sqrt{-E}\sigma(t)$*, the functions*

$$X(t) = a(t)e^{\alpha(t)} \quad \text{and} \quad Y(t) = a(t)e^{-\alpha(t)/2}$$

solve the Bianchi I field equations.

A similar result is currently being investigated for the more general metric

$$ds^2 = -dt^2 + X(t)dx^2 + Y(t)^2dy^2 + Z(t)^2dz^2$$

References

[1] D. Bazeia, *Scalar fields: from domain walls to nanotubes and fulerenes* arXiv:hep-th/0203050.

[2] J. D'Ambroise, F. Williams, *A non-linear Schrodinger type formulation of FLRW scalar field cosmology,* arXiv.org hep-th/0609125.

[3] G. Ellis, M. Madsen, *Exact scalar field cosmologies, Classical and Quantum Gravity,* **8** (1991) 667-676.

[4] R. Hawkins, J. Lidsey, *The Ermakov-Pinney equation in scalar field cosmologies, Physical Review D,* **66** (2002), 023523-023531.

[5] P. Kevrekidis, F. Williams, *On 2+1-dimensional Friedmann-Robertson-Walker universes: an Ermakov-Pinney equation approach, Classical and Quantum Gravity*, **20** (2003) L177-L184.

[6] J. Lidsey, *Cosmic dynamics of Bose-Einstein Condensates, Classical and Quantum Gravity*, **21** (2004), 777-785.