SMALL EXOTIC 4–MANIFOLDS AND SYMPLECTIC CALABI-YAU SURFACES VIA GENUS–3 PENCILS

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Abstract. In this article, we introduce a strategy to produce exotic rational and elliptic ruled surfaces, and possibly new symplectic Calabi-Yau surfaces, via constructions of symplectic Lefschetz pencils using a novel technique we call breeding. We deploy our strategy to breed explicit symplectic genus–3 pencils, whose total spaces are homeomorphic but not diffeomorphic to the rational surfaces $\mathbb{CP}^2 \# p \overline{\mathbb{CP}}^2$ for $p = 6, 7, 8, 9$. Similarly, we breed explicit genus–3 pencils, whose total spaces are symplectic Calabi-Yau surfaces that have $b_1 > 0$ and realize all the integral homology classes of torus bundles over tori.

1. Introduction

Since the advent of Gauge theory, many construction techniques, such as knot surgery, rational blowdowns, generalized fiber sums and Luttinger surgery, have been introduced and successfully employed to produce exotic smooth structures on 4–manifolds, primarily through constructions of symplectic 4–manifolds homeomorphic but not diffeomorphic to smooth connected sums of standard 4–manifolds, where those with small topology (i.e. small second homology) have proven to be the most challenging. (e.g. [1, 2, 3, 7, 15, 20, 28, 29, 33, 32, 37, 40, 49, 52, 60, 62, 63, 67, 70].)

In this article we deploy a strategy to produce small symplectic 4–manifolds as total spaces of Lefschetz pencils,[1] which correspond to *small positive factorizations* (i.e. small number of Dehn twists) we construct using a new technique we will discuss below. Recall that by the celebrated work of Donaldson [21] any compact symplectic 4–manifold admits a Lefschetz pencil, and in turn, corresponds to a positive factorization in the mapping class group of an orientable surface [50, 61, 58]. For the small 4–manifolds we consider, the additional information presented by the pencil structure will be crucial to detect the exotic smooth structures, as illustrated by our first theorem:

Theorem A. *Let X be a symplectic 4–manifold homeomorphic to a rational or ruled surface Z with $c_1^2(Z) \geq 0$. Then X is an exotic Z if and only if it admits a genus–g Lefschetz pencil with number of base points $b \leq 2g - 2 - \chi_h(Z)$.*

[1]Conventions: We assume that Lefschetz *pencils*, unlike Lefschetz fibrations, always have base points, whereas both have critical points and no exceptional spheres contained in the fibers.

Here $c_1^2 = 2\mathrm{e} + 3\,\sigma$ is the first Chern number and $\chi_h = \frac{1}{4}(\mathrm{e} + \sigma)$ is the holomorphic Euler characteristic, where e and σ are the Euler characteristic and the signature of the 4–manifold. Rational and ruled surfaces satisfying the $c_1^2 \geq 0$ condition are the rational surfaces $\mathbb{CP}^2 \# \, p\, \overline{\mathbb{CP}}{}^2$, for $p \leq 9$, and $S^2 \times S^2$ (which have $\chi_h = 1$), and the minimal elliptic ruled surfaces $T^2 \times S^2$ and $T^2 \,\tilde{\times}\, S^2$ (which have $\chi_h = 0$). The existence of the Lefschetz pencils in the statement of the theorem are granted by Donaldson, whereas our proof of the essential constraints on the topology of the pencils uses Seiberg–Witten theory, and builds on the works of Taubes [72, 73], McDuff [59] and Li–Liu [57]. We note that while there are numerous constructions of minimal symplectic 4–manifolds homeomorphic but not diffeomorphic to the rational surfaces $\mathbb{CP}^2 \# \, p\, \overline{\mathbb{CP}}{}^2$, for $p \geq 2$ (see all the reference listed above), there are no known examples of exotic irrational ruled surfaces to date.

The homeomorphism types of the rational and elliptic ruled surfaces are easily determined by their fundamental group and intersection form by Freedman [34], and Hambleton–Kreck [45], respectively. Thus, powered by Theorem A, one can produce exotic copies of these small 4–manifolds by constructing Lefschetz pencils with the right algebraic invariants and small number of base points relative to the fiber genus. As a successful implementation of this approach, we show that:

Theorem B. *There are symplectic genus–3 Lefschetz pencils $\{(X_{i,\phi}, f_{i,\phi})\}$ whose total spaces have $\chi_h(X_{i,\phi}) = 1$ and $c_1^2(X_{i,\phi}) = 3 - i$, and they include exotic rational surfaces $\mathbb{CP}^2 \#(6+i)\overline{\mathbb{CP}}{}^2$ as well as infinitely many symplectic 4–manifolds which are not homotopy equivalent to any complex surface, for each $i = 0, 1, 2, 3$.*

The index ϕ for the family of pencils $\{(X_{i,\phi}, f_{i,\phi})\}$ takes values in a certain infinite subgroup of the mapping class group $\mathrm{Mod}(\Sigma_3^1)$ for each $i = 0, 1, 2, 3$.

Each symplectic 4–manifold $X_{i,\phi}$ in the theorem is "almost minimal", that is, it is either minimal or at most once blow-up of a minimal symplectic 4–manifold; see Remark 8. Notably, our family of genus–3 pencils with $c_1^2 = 3$ are all hyperelliptic, and therefore, by the work of Siebert-Tian [66], each $X_{0,\phi}$, including our exotic $\mathbb{CP}^2 \# 6\, \overline{\mathbb{CP}}{}^2$, admits a symplectic involution and is a blow-down of a symplectic double branched covering of a rational surface; see Remark 10. We should also note that $g = 3$ is the smallest fiber genus for any Lefschetz *pencil* on an exotic rational surface, and we moreover suspect that our examples in Theorem B are also optimal in regard to the smallest exotic rational surfaces one can obtain via genus–3 pencils; see Remark 9. While in this article we only study genus $g = 3$ pencils, one can obtain much sharper results even with $g = 4$ or 5 pencils, as demonstrated in our forthcoming work in [12].

We describe our Lefschetz pencils in Theorem B in terms of their monodromy factorizations, which amount to positive Dehn twist factorizations of the boundary multi-twist in the mapping class group of an orientable surface. We build these pencils out of lower genera pencils, using a novel technique we call *breeding*, which consists of carefully embedding the positive factorizations for lower genera pencils into the mapping class group of a higher genus surface in a way that one can cancel all the negative Dehn twists (along non-boundary parallel curves) against positive ones at the end. It is worth noting that, although we use the breeding technique to derive new symplectic 4–manifolds from smaller ones, it is not an inherently symplectic operation. In the intermediary steps we get achiral Lefschetz pencils and fibrations which do contain negative nodes, but then we match them with positive nodes and remove all these pairs —which corresponds to surgering out self-intersection zero spheres contained in the fibers.

A simpler version of the breeding technique was used by Korkmaz and the author (in an unpublished note) to produce hyperelliptic genus–g Lefschetz fibrations with $5g-3$ critical points, which yield the smallest hyperelliptic Lefschetz fibrations when $g=3$. Since the appearance of the first version of this paper on the arxiv, the breeding technique has been used to produce several new Lefschetz pencils and fibrations (e.g. [43, 4, 11, 12]) and especially played a vital role in the recent resolution of Stipsicz's conjecture on the signature of Lefschetz fibrations in [11].

In the last portion of our paper, we turn to symplectic Calabi-Yau surfaces. Recall that a symplectic 4–manifold is called a *symplectic Calabi-Yau surface* if it has trivial canonical class, in obvious analogy with complex Calabi-Yau surfaces. The works of T.-J. Li and Bauer established that any symplectic Calabi-Yau surface with $b_1>0$ has the rational homology type of a torus bundle over torus [53, 55, 8], and it remains an open question whether torus bundles over tori exhaust all the diffeomorphism types of symplectic Calabi-Yau surfaces with $b_1>0$ [53, 23]. As stated by T.-J. Li [56], a posteriori reasoning for an affirmative answer to this question often seems to stem from the lack of any new constructions of symplectic Calabi-Yau surfaces. The surgical operations like knot surgery, simplest rational blow-downs, generalized fiber sums or Luttinger surgery, do not produce any new symplectic Calabi-Yau surfaces [47, 56, 75, 24].

Akin to our strategy for producing exotic rational and elliptic ruled surfaces, in [13, 14] we implemented a strategy to construct (possibly new) symplectic Calabi-Yau surfaces via positive factorizations for pencils. The breeding technique, which is particularly effective for getting small positive factorizations, allows us to produce small symplectic Calabi-Yau surfaces as well:

Theorem C. *There are symplectic genus–3 Lefschetz pencils $\{(X_\phi, f_\phi)\}$ whose total spaces are symplectic Calabi-Yau surfaces that realize all integral homology types of torus bundles over tori, and they include a symplectic Calabi-Yau surface homeomorphic to the 4–torus and fake symplectic $T^2\times S^2$s.*

The index ϕ for the family of pencils $\{(X_\phi, f_\phi)\}$ takes values in a certain infinite subgroup of the mapping class group $\mathrm{Mod}(\Sigma_3^4)$. A fake $T^2\times S^2$ is a 4–manifold which has the same homology type as $T^2\times S^2$ but is not diffeomorphic to it.

We describe the Lefschetz pencils in Theorem C in terms of their monodromy factorizations given in the equation (47), which feeds into Donaldson's proposal of analyzing monodromies of pencils on symplectic Calabi-Yau surfaces [22][Problem 5]. These are the first explicit monodromy factorizations of pencils on symplectic Calabi-Yau surfaces with $b_1 > 0$ in the literature, whereas many examples on symplectic Calabi-Yau surfaces with $b_1 = 0$ were obtained in [13, 14]. Following the publicizing of an earlier version of this paper, similar examples were obtained by Hamada and Hayano in [43] also by employing the breeding technique.

Since symplectic Calabi-Yau surfaces with $b_1 > 0$ have the same Seiberg-Witten invariants as torus bundles over tori, detecting any new symplectic Calabi-Yau surfaces among $\{X_\phi\}$ hangs on essentially the possibility of detecting a $\pi_1(X_\phi)$ that is not a torus bundle group; see Remark 14. At the time of writing, we have not been able to determine whether all $\pi_1(X_\phi)$ we get are torus bundle groups. Likewise, we have not been able to spot any fake symplectic $T^2 \times S^2$ among $\{X_\phi\}$ with $\pi_1(X_\phi) = \mathbb{Z}^2$, which would make it homeomorphic to $T^2 \times S^2$, and thus an exotic elliptic ruled surface. (There are torus bundle over tori which have the same homology type as $T^2 \times S^2$.) On the other hand, Hamada and Hayano were able to show in [43] that our symplectic Calabi-Yau surface homeomorphic to the 4–torus is in fact diffeomorphic to it, by comparing our example with a holomorphic pencil on the standard 4–torus described by Smith; see Remark 15. While we do not know if any other X_ϕ is standard, it is worth noting that if our family of symplectic Calabi-Yau surfaces $\{X_\phi\}$ were to fully overlap with torus bundles over tori, then an additional feature of our construction would imply that any of these bundles can be equipped with a symplectic structure so that it is obtained via Luttinger surgeries from the standard 4–torus [47][Conjecture 4.9]; see Remark 16.

Outline of the paper: We review the basic definitions and preliminary results on Lefschetz pencils and fibrations, mapping class groups and positive factorizations, and symplectic 4–manifolds and Calabi-Yau surfaces in Section 2. In Section 3, we provide a characterization of small symplectic exotic rational surfaces (Theorem 3) and that of exotic minimal ruled surfaces (Theorem 6), which together give Theorem A. We breed our genus–3 pencils on exotic rational surfaces in Section 4, and on symplectic Calabi-Yau surfaces with $b_1 > 0$ in Section 5, which yield Theorems B and C, respectively.

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2. Preliminaries

Here we quickly review the definitions and the basic properties of Lefschetz pencils and fibrations, Dehn twist factorizations in mapping class groups of surfaces, and symplectic 4–manifolds. The reader can turn to [41, 54, 15] for more details.

2.1. **Lefschetz pencils and fibrations.**

A *Lefschetz pencil* on a closed, smooth, oriented 4-manifold X is a smooth surjective map $f : X \setminus \{b_j\} \to S^2$, defined on the complement of a non-empty finite collection of points $\{b_j\}$, such that around every *base point* b_j and *critical point* p_i there are local complex coordinates (compatible with the orientations on X and S^2) with respect to which the map f takes the form $(z_1, z_2) \mapsto z_1/z_2$ and $(z_1, z_2) \mapsto z_1 z_2$, respectively. A *Lefschetz fibration* is defined similarly for $\{b_j\} = \emptyset$. Blowing-up at each base point b_j of a pencil (X, f), one obtains a Lefschetz fibration $(\widetilde{X}, \widetilde{f})$ with disjoint (-1)–sphere sections S_j corresponding to each b_j, and conversely, blowing down disjoint (-1)–sphere sections of a Lefschetz fibration, one obtains a pencil.

We say (X, f) is a *genus–g Lefschetz pencil* or *fibration* for g the genus of a *regular fiber* F of f. The fiber containing the critical point p_i has a *nodal singularity* at p_i, which locally arises from shrinking a simple loop c_i on F, called a *vanishing cycle*. A singular fiber of (X, f) is called *reducible* if c_i is separating. When c_i is null-homotopic on F, one of the fiber components becomes an *exceptional sphere*, an embedded 2–sphere of self-intersection -1, which one can blow down without altering the rest of the fibration.

In this paper we use the term Lefschetz fibration only when the set of critical points $\{p_i\}$ is non-empty, i.e. when the Lefschetz fibration is *non-trivial*. We moreover assume that the fibration is *relatively minimal*, i.e. there are no exceptional spheres contained in the fibers, and also that the critical points p_i lie in distinct *singular fibers*, which can be always achieved after a small perturbation.

Allowing the local model $(z_1, z_2) \mapsto z_1 \bar{z}_2$ around the critical points p_i, which give rise to *negative* nodes, all of the above notions extend to so-called *achiral* Lefschetz pencils and fibrations.

2.2. **Positive factorizations.**

Let Σ_g^b denote a compact, connected, oriented surface genus g with b boundary components, and simply write Σ_g when there is no boundary. We denote by $\mathrm{Mod}(\Sigma_g^b)$ its *mapping class group*; the group composed of orientation-preserving self-homeomorphisms of Σ_g^m which restrict to the identity along $\partial\Sigma_g^b$, modulo isotopies that also restrict to the identity along $\partial\Sigma_g^b$. Let $\mathrm{Mod}(\Sigma_g^b, S)$ denote the stabilizer subgroup of $\mathrm{Mod}(\Sigma_g^b)$ which consists of elements fixing the subset $S \subset \Sigma_g^b$ pointwise. Denote by $t_c \in \mathrm{Mod}(\Sigma_g^b)$ the positive (right-handed) Dehn twist along the simple closed curve $c \subset \Sigma_g^m$. Its inverse t_c^{-1} is the negative (left-handed) Dehn twist along c.

Let $\{c_i\}$ be a *non-empty* collection of simple closed curves on Σ_g^b, which do not become null-homotopic when $\partial\Sigma_g^b$ is capped off by disks, and let $\{\delta_j\}$ be a collection of curves parallel to distinct boundary components of Σ_g^b. If the relation

$$t_{c_l} \cdots t_{c_2} t_{c_1} = t_{\delta_1} \cdots t_{\delta_b} \tag{1}$$

holds in $\mathrm{Mod}(\Sigma_g^b)$, we call the word W on the left-hand side a *positive factorization* of the *boundary multi-twist* $\Delta = t_{\delta_1} \cdots t_{\delta_b}$ in Γ_g^b. (We will also use ∂_i instead of δ_i when there are several surfaces with boundaries involved in our discussion.) Capping off all the boundary components of Σ_g^b with disks induces a homomorphism $\mathrm{Mod}(\Sigma_g^b) \to \mathrm{Mod}(\Sigma_g)$, under which W maps to a similar positive factorization of the identity element $1 \in \mathrm{Mod}(\Sigma_g)$.

The positive factorization in (1) gives rise to a genus–g Lefschetz fibration $(\widetilde{X}, \widetilde{f})$ with b disjoint (-1)–sections S_j, and therefore a genus–g Lefschetz pencil (X, f) with b base points. Identifying the regular fiber F with Σ_g, we can view the vanishing cycles of the fibration as the Dehn twist curves $\{c_i\}$. Every Lefschetz pencil and fibration can be described by such a positive factorization, which is called its *monodromy factorization* [50, 58, 41].

Let W be a positive factorization of the form $W = PP'$ in $\mathrm{Mod}(\Sigma_g^b)$, where P and P' are some products of positive Dehn twists along curves which do not become null-homotopic when $\partial\Sigma_g^b$ is capped off. If $P = \Pi_i\, t_{c_i}$, as a mapping class, commutes with some element $\phi \in \mathrm{Mod}(\Sigma_g^b)$, we can then produce a new positive factorization $W_\phi = P^\phi P'$, where P^ϕ denotes the conjugate factorization $\phi P \phi^{-1} = \Pi_i\,(\phi\, t_{c_i}\, \phi^{-1}) = \Pi_i\, t_{\phi(c_i)}$. In this case, we say W_ϕ is obtained from W by a *partial conjugation ϕ along P*.

Allowing *negative* Dehn twists, which correspond to negative nodes, we can more generally work with factorizations for achiral Lefschetz fibrations and pencils. All of the above definitions and results extend to this more general setting.

2.3. **Symplectic 4-manifolds and Kodaira dimension.**

It was shown by Donaldson that every symplectic 4-manifold (X, ω) admits a *symplectic* Lefschetz pencil whose fibers are symplectic with respect to ω [21]. Conversely, generalizing a construction of Thurston, Gompf showed that the total space of a Lefschetz pencil and fibration always admits a symplectic form ω with respect to which all regular fibers and any preselected collection of disjoint sections are symplectic [41]. Whenever we take a symplectic form ω on a Lefschetz pencil or fibration (X, f), we will assume it is of Thurston-Gompf type, with respect to which any explicitly mentioned sections will be assumed to be symplectic as well.

The Kodaira dimension for projective surfaces can be extended to symplectic 4–manifolds as follows: Let $K_{X_{\min}}$ be the canonical class of a minimal model $(X_{\min}, \omega_{\min})$ of (X, ω). The *symplectic Kodaira dimension* of (X, ω), denoted by $\kappa = \kappa(X, \omega)$ is then defined as

$$\kappa(X,\omega) = \begin{cases} -\infty & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] < 0 \;\text{ or }\; K^2_{X_{\min}} < 0 \\ 0 & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] = K^2_{X_{\min}} = 0 \\ 1 & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] > 0 \;\text{ and }\; K^2_{X_{\min}} = 0 \\ 2 & \text{if } K_{X_{\min}} \cdot [\omega_{\min}] > 0 \;\text{ and }\; K^2_{X_{\min}} > 0. \end{cases}$$

Remarkably, not only κ is independent of the minimal model $(X_{\min}, \omega_{\min})$ but also it is independent of the particular symplectic form ω on X; so it is a smooth invariant of the 4–manifold X [53]. Symplectic 4–manifolds with $\kappa = -\infty$ are classified up to symplectomorphisms, which are precisely the rational and ruled surfaces [54].

Symplectic 4–manifolds with $\kappa = 0$, which are the analogues of the Calabi-Yau surfaces, are those with torsion canonical class [53]. It was shown by Tian-Jun Li, and independently by Stefan Bauer [53, 8], that the rational homology type of any *minimal* symplectic 4–manifold with $\kappa = 0$ is that of a torus bundle over a torus, the K3 surface or the Enrique surfaces. In the first two cases we have *symplectic Calabi-Yau surfaces*, which have trivial canonical class, whereas in the last case the canonical class is torsion.

We have the following topological characterization of Lefschetz pencils on *minimal* symplectic 4–manifolds with $\kappa = 0$, which can be easily derived from the more general characterization for Lefschetz fibrations on symplectic 4–manifolds with $\kappa = 0$ given in [13, Theorem 4.1], [65, Theorem 5.12]:

Proposition 1. *Let (X, f) be a genus–g Lefschetz pencil with b base points, where X is neither rational nor ruled. Then there is a symplectic form ω on X so that (X, ω) is a symplectic Calabi-Yau or a rational homology Enriques surface if and only if $b = 2g - 2$.*

3. Topology of pencils on rational and elliptic ruled surfaces

In this section we will prove two theorems that might be of independent interest; one on the topology of Lefschetz pencils and fibrations on (small) rational surfaces, and one on (small) irrational ruled surfaces. These results enable one to tackle producing exotic smooth structures on the rational surfaces $\mathbb{CP}^2 \# \, p\, \overline{\mathbb{CP}}^2$, $S^2 \times S^2$, and the minimal elliptic ruled surfaces $T^2 \times S^2$ and $T^2 \tilde{\times} S^2$, via constructions of new positive factorizations, as we will try to demonstrate in the later sections.

3.1. Lefschetz pencils and fibrations on rational surfaces.

We first prove the following lemma, which shows that pencils on rational surfaces always have a lot of base points with respect to the fiber genera:

Lemma 2. *The rational surfaces $\mathbb{CP}^2 \# p\,\overline{\mathbb{CP}}^2$, for $p \leq 9$, or $S^2 \times S^2$, do not admit any genus–g pencil with $b < 2g-2$ base points or any genus $g \geq 2$ Lefschetz fibration.*

Proof. We claim that the statement of the lemma holds even for non-relatively minimal pencils and fibrations. With this in mind, it suffices to prove our claim for $X = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$, because we can blow-up on the fibers of a given genus–g Lefschetz pencil or fibration on $\mathbb{CP}^2 \# p\overline{\mathbb{CP}}^2$ with $p < 9$ or $S^2 \times S^2$ to get one on X.

Now suppose for contradiction that $X = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$ admits a genus–g pencil with $b < 2g-2$ base points or a genus $g \geq 2$ Lefschetz fibration. Note that in either case $g \geq 2$. For our arguments to follow, it will be convenient to allow b to be a non-negative integer so that $b = 0$ marks the fibration case.

Let $F = aH - \sum_{i=1}^{9} c_i E_i$ be the fiber class, where $H_2(X)$ is generated by the hyperplane class H and the exceptional classes $E_1, \ldots, E_9$, which satisfy $H^2 = 1$, $E_i \cdot E_j = -\delta_{ij}$, and $H \cdot E_i = 0$. Since $F^2 = b$, we have

$$a^2 = b + \sum_{i=1}^{9} c_i^2 \,.$$

We can equip X with a Thurston-Gompf symplectic form ω which makes the fibers symplectic. Moreover, we can choose an ω-compatible almost complex structure J, even a generic one in the sense of Taubes, with respect to which the pencil/fibration is J-holomorphic for a suitable choice of almost complex structure on the base 2-sphere; see e.g [74]. It was shown by Li and Liu [57] that for a generic ω-compatible J, the class H in the rational surface X has a J-holomorphic representative . Hence, F and H both have J-holomorphic representatives, which implies that $F \cdot H = a \geq 0$.

Since there is a unique symplectic structure on X up to deformation and symplectomorphisms [57], we can apply the adjunction formula to get

$$2g-2 = F^2 + K \cdot F = b + (-3H + \sum_{i=1}^{9} E_i) \cdot (aH - \sum_{i=1}^{9} c_i E_i) = b - 3a + \sum_{i=1}^{9} c_i \,.$$

Since $a, b \geq 0$, and $g \geq 2$, from the above equalities we have

$$3a = \sqrt{9a^2} = \sqrt{9(b + \sum_{i=1}^{9} c_i^2)} \geq \sqrt{9(\sum_{i=1}^{9} c_i^2)} = \sqrt{(\sum_{i=1}^{9} 1)(\sum_{i=1}^{9} c_i^2)} \geq \sqrt{|\sum_{i=1}^{9} c_i|^2} \,,$$

where the last inequality is by Cauchy-Schwartz. In turn, we get:

$$3a \geq \sqrt{|\sum_{i=1}^{9} c_i|^2} = |\sum_{i=1}^{9} c_i| = |2g-2-b+3a| = 2g-2-b+3a \,,$$

which implies that $b \geq 2g-2$. The contradiction shows that there is no such fiber class F. In turn, there is no such Lefschetz pencil or fibration. $\square$

The statement as stated is obviously not true for $p > 9$; for example there is a genus–2 Lefschetz fibration on $\mathbb{CP}^2 \# 13\,\overline{\mathbb{CP}}^2$ (which in fact is the blow-up of a genus–2 pencil on $S^2 \times S^2$). Otherwise, one can generalize the above result to rational surfaces $\mathbb{CP}^2 \# p\,\overline{\mathbb{CP}}^2$ with $p > 9$, under particular assumptions for b and g with respect to the number of blow-ups p.

Since by Donaldson any symplectic 4–manifold admits a Lefschetz pencil, and since only rational surfaces admit genus–0 or genus–1 Lefschetz pencils and fibrations, we can then conclude that:

Theorem 3. *A symplectic 4-manifold X in the homeomorphism class of $\mathbb{CP}^2 \# \, p\,\overline{\mathbb{CP}}^2$ with $p \leq 9$ or $S^2 \times S^2$ is an exotic rational surface if and only if it admits a genus–g pencil with $b < 2g - 2$ base points or a genus $g \geq 2$ Lefschetz fibration.*

As we mentioned earlier, there are numerous examples of symplectic 4–manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# \, p\,\overline{\mathbb{CP}}^2$, for $2 \leq p \leq 9$, and they should all admit genus–g pencils with $b < 2g - 2$ base points by the above theorem. However, in the literature there appears to be no examples of Lefschetz pencils (with base points, no multiple fibers) on these 4–manifolds, even on the complex algebraic ones. We will provide some novel symplectic examples admitting genus–3 pencils in the next section.

As for fibrations, for K any genus $g \geq 1$ fibered knot, knot surgered elliptic surfaces $E(1)_K$ of Fintushel and Stern, yield exotic $E(1) = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$, which admit symplectic genus–$2g$ Lefschetz fibrations [30]. Moreover, there are genus–2 symplectic Lefschetz fibrations in the homeomorphism classes of $\mathbb{CP}^2 \# \, p\,\overline{\mathbb{CP}}^2$ for $p = 7, 8, 9$ [15] and even holomorphic ones for $p = 8, 9$ [64].

Remark 4. When X is an exotic $\mathbb{CP}^2 \# \, p\,\overline{\mathbb{CP}}^2$, with $p \leq 8$, we can strengthen the statement of Theorem 3 a bit. If the genus $g \geq 2$ pencil on X had $b = 2g - 3$ base points, blowing up all of them, we would get a Lefschetz fibration with b disjoint (-1)–sphere sections. It then follows from [65, Theorem 5-12] that $K^2_{X_{\min}} = 0$, which cannot be the case here since $K^2_{X_{\min}} \geq K^2_X = 9 - p > 0$. Hence, any pencil on such an exotic rational surface X can have at most $2g - 4$ base points.

3.2. **Lefschetz pencils and fibrations on minimal elliptic ruled surfaces.**

We now show that pencils on minimal elliptic ruled surfaces also have a lot of base points with respect to the fiber genera:

Lemma 5. *The minimal elliptic ruled surfaces $T^2 \times S^2$ or $T^2 \tilde{\times} S^2$ do not admit any genus–g Lefschetz pencil with $b \leq 2g - 2$ base points or any Lefschetz fibration.*

Proof. These minimal elliptic surfaces do not admit genus $g < 2$ Lefschetz fibrations for fairly elementary reasons (which do not require classification results): any genus–0 Lefschetz fibration has a simply-connected total space, and the Euler characteristic of any genus–1 Lefschetz fibration is equal to the number of critical points, and therefore it is positive. Clearly, neither one of these two implications work for $T^2 \times S^2$ or $T^2 \tilde{\times} S^2$.

Now suppose for a contradiction that $X = T^2 \times S^2$ or $T^2 \tilde{\times} S^2$ admits a genus–g pencil with $b \leq 2g - 2$ base points or a Lefschetz fibration. Once again, it will be convenient here to let b be a non-negative integer so that $b = 0$ marks the fibration case. By our observation in the previous paragraph, we can assume that $g \geq 2$.

We equip X with a Thurston-Gompf symplectic form ω which makes the fibers, and in particular a regular fiber F, of the pencil/fibration symplectic. We can choose an ω–compatible almost complex structure J with respect to which the pencil/fibration is J-holomorphic, so in particular F is a J–holomorphic curve. Because there is a unique symplectic structure on a minimal ruled surface up to deformations and symplectomorphisms [57], we will be able to once again apply the adjunction formula using a standard canonical class in each case. Furthermore, it will be important for our arguments that it was also shown in [57] that for any ω-compatible almost complex structure J, the sphere fiber of the ruling on the elliptic surface has a J–holomorphic representative. Therefore the algebraic intersection of F with the sphere fiber is non-negative. Akin to our proof of Lemma 2, we will show that neither one of the minimal elliptic ruled surfaces contains an embedded genus–g symplectic surface with self-intersection $\leq 2g - 2$, whereas F is such.

We will run our arguments for the spin and non-spin cases separately:

$\underline{X = T^2 \times S^2}$: Here $H_2(X) \cong \mathbb{Z}^2$ is generated by $S = \{pt\} \times S^2$ and $T = T^2 \times \{pt\}$, where $S \cdot S = 0$, $T \cdot T = 0$, and $S \cdot T = 1$. By a slight abuse of notation, we denote the homology class of the fiber also by F, so $F = x\, S + y\, T$ for some $x, y \in \mathbb{Z}$.

As remarked above, the algebraic intersection of F with S is non-negative, which means that $F \cdot S = y \geq 0$. Since $F^2 = b$, we have

$$b = 2\, xy\,,$$

where $b \geq 0$ and $y \geq 0$ imply that $x \geq 0$.

On the other hand, by the adjunction formula we get

$$2g - 2 = F^2 + K_X \cdot F = b + (-2T) \cdot (x\, S + y\, T) = b - 2x\,,$$

which implies that $2x = b - (2g - 2) \leq 0$ by our assumption on b. It follows that $x = 0$, and in turn, $b = 0$ by the first equality, and $g = 1$ by the second, which is a contradiction.

$\underline{X = T^2 \tilde{\times} S^2}$: Now $H_2(X) \cong \mathbb{Z}^2$ is generated by the fiber S and section T of the degree–1 ruling on X, where $S \cdot S = 0$, $T \cdot T = 1$, and $S \cdot T = 1$. Let the fiber class F be given by $F = x\, S + y\, T$, for some $x, y \in \mathbb{Z}$.

Since the algebraic intersection of F with S is non-negative, we have $F \cdot S = y \geq 0$. Since $F^2 = b$, we now have

$$b = 2\,xy + y^2 = (2x + y)\,y\,,$$

where $b \geq 0$ and $y \geq 0$ imply that $2x + y \geq 0$.

We apply the adjunction formula to get

$$2g - 2 = F^2 + K_X \cdot F = b + (S - 2T) \cdot (x\,S + y\,T) = b - 2x - y\,,$$

which means that $2x + y = b - (2g - 2) \leq 0$. As we also had $2x + y \geq 0$, it follows that $2x + y = 0$, and then $b = 0$ by the first equality, and $g = 1$ by the second, which once again contradicts our assumption on the fiber genus —–that $g \geq 2$. □

The above result, at least as stated, does not generalize to pencils on other ruled surfaces. First of all, there exist genus–g Lefschetz pencils with $b = 2g-2$ base points on non-minimal elliptic ruled surfaces, even after a single blow-up; an example with $g = b = 2$ can be found in the next section. Secondly, there are pencils on the minimal ruled surfaces $\Sigma_h \times S^2$ and $\Sigma_h \tilde{\times} S^2$ with fiber genus $g = 2h$ and $b = 4$ base points [42], so the statement fails for any $h \geq 2$ in both spin and non-spin cases.

Using Donaldson's result on the existence of Lefschetz pencils on symplectic 4–manifolds, we moreover conclude that:

Theorem 6. *A symplectic 4-manifold X in the homeomorphism class of $T^2 \times S^2$ or $T^2 \tilde{\times} S^2$ is an exotic elliptic ruled surface if and only if it admits a genus–g pencil with $b \leq 2g - 2$ base points or a Lefschetz fibration.*

It is worth noting that to this date there are no known examples of exotic elliptic ruled surfaces, despite their topological types being amenable to Freedman type arguments [46]. While we plan to explore this direction elsewhere, in Section 5, through positive factorizations for pencils, we will provide examples of *fake* symplectic elliptic ruled surfaces, which have the same cohomology as $T^2 \times S^2$, but are not diffeomorphic to it.

4. Exotic rational surfaces via symplectic genus-3 pencils

Here we construct positive factorizations for symplectic genus–3 Lefschetz pencils, whose total spaces are homeomorphic but not diffeomorphic to rational surfaces. These will be bred from genus–2 pencils on elliptic ruled surfaces. For a better exposition, we first present our examples with $\chi_h = 1$ and $c_1^2 = 0, 1, 2$, and we discuss our examples with $\chi_h = 1$ and $c_1^2 = 3$, whose constructions are a bit more involved, afterwards.

4.1. **Breeding pencils with $\chi_h = 1$ and $c_1^2 = 0, 1, 2$.**

In [15], Korkmaz and the author obtained the following relation in $\mathrm{Mod}(\Sigma_2^1)$:

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_C t_{x_4} = t_\delta \,,$$

which is a positive factorization for a genus–2 pencil with one base point on $T^2 \times S^2 \# 2\overline{\mathbb{CP}}{}^2$. See Figure 2 for the curves x_i, C, d, e (where the boundary component δ is obtained by carving out a disk neighborhood of the marked point on right end of the surface). Consider the embedding of Σ_2^1 into Σ_3^1 given by mapping the boundary $\delta = \partial \Sigma_2^1$ to the curve C', and the remaining Dehn twist curves x_i, C, d, e to the ones shown in Figure 1, denoted by the same letters. After a single Hurwitz move, and collecting all the Dehn twists on the same side, we get the following relation in $\mathrm{Mod}(\Sigma_3^1)$:

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_C t_{C'}^{-1} = 1 \,, \tag{2}$$

where $B_2 = t_C(x_4)$. Rewrite this relation as $P_1 \, t_C t_{C'}^{-1} = 1$, for $\underline{P_1 = t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2}}$. Note that P_1, t_C and $t_{C'}$ all commute with each other.

Next, we take the following lift of the positive factorization for Matsumoto's genus–2 Lefschetz fibration to $\mathrm{Mod}(\Sigma_2^2)$ obtained by Hamada in [42]:

$$(t_{B_0} t_{B_1} t_{B_2} t_C)^2 = t_{\delta_1} t_{\delta_2} \,,$$

where δ_i are the boundary parallel curves, and the curves B_i and C are as shown on the left-hand side of Figure 6. This is a positive factorization for a genus–2 pencil with two base points on $T^2 \times S^2 \# 2\overline{\mathbb{CP}}{}^2$. After Hurwitz moves, and collecting all the Dehn twists on the same side, we get the following relation in $\mathrm{Mod}(\Sigma_2^2)$:

$$t_{B_0} t_{B_1} t_{B_2} t_{A_0} t_{A_1} t_{A_2} t_C^2 t_{\delta_1}^{-1} t_{\delta_2}^{-1} = 1 \,,$$

where each $A_j = t_C(B_j)$, for $j = 0, 1, 2$, are as shown in Figure 6. We will describe two different embeddings of this relation into $\mathrm{Mod}(\Sigma_3^1)$.

Cap off the boundary component δ_1 of Σ_2^2, and then embed the resulting copy of Σ_2^1 into Σ_3^1 via the embedding we used to derive the relation (2) above, so the boundary δ_2 is mapped to C', and all the other Dehn twist curves are as shown in Figure 1, once again denoted by the same letters. So we have the following relation in $\mathrm{Mod}(\Sigma_3^1)$:

$$t_{B_0} t_{B_1} t_{B_2} t_{A_0} t_{A_1} t_{A_2} t_C^2 t_{C'}^{-1} = 1 \,, \tag{3}$$

which we rewrite as $P_2 \, t_C^2 t_{C'}^{-1} = 1$, for $\underline{P_2 = t_{B_0} t_{B_1} t_{B_2} t_{A_0} t_{A_1} t_{A_2}}$. Here P_2, t_C and $t_{C'}$ all commute with each other.

Lastly, consider an embedding of Σ_2^2 into Σ_3^1 so that δ_1 is mapped to c, δ_2 is mapped to $\partial = \partial \Sigma_3^1$, and the remaining curves are as shown in Figure 1, where we use a prime symbol when denoting the curves by the same letters. This gives a third relation in $\mathrm{Mod}(\Sigma_3^1)$:

$$t_{B'_0} t_{B'_1} t_{B'_2} t_{A'_0} t_{A'_1} t_{A'_2} t_{C'}^2 t_C^{-1} = t_\partial \,, \tag{4}$$

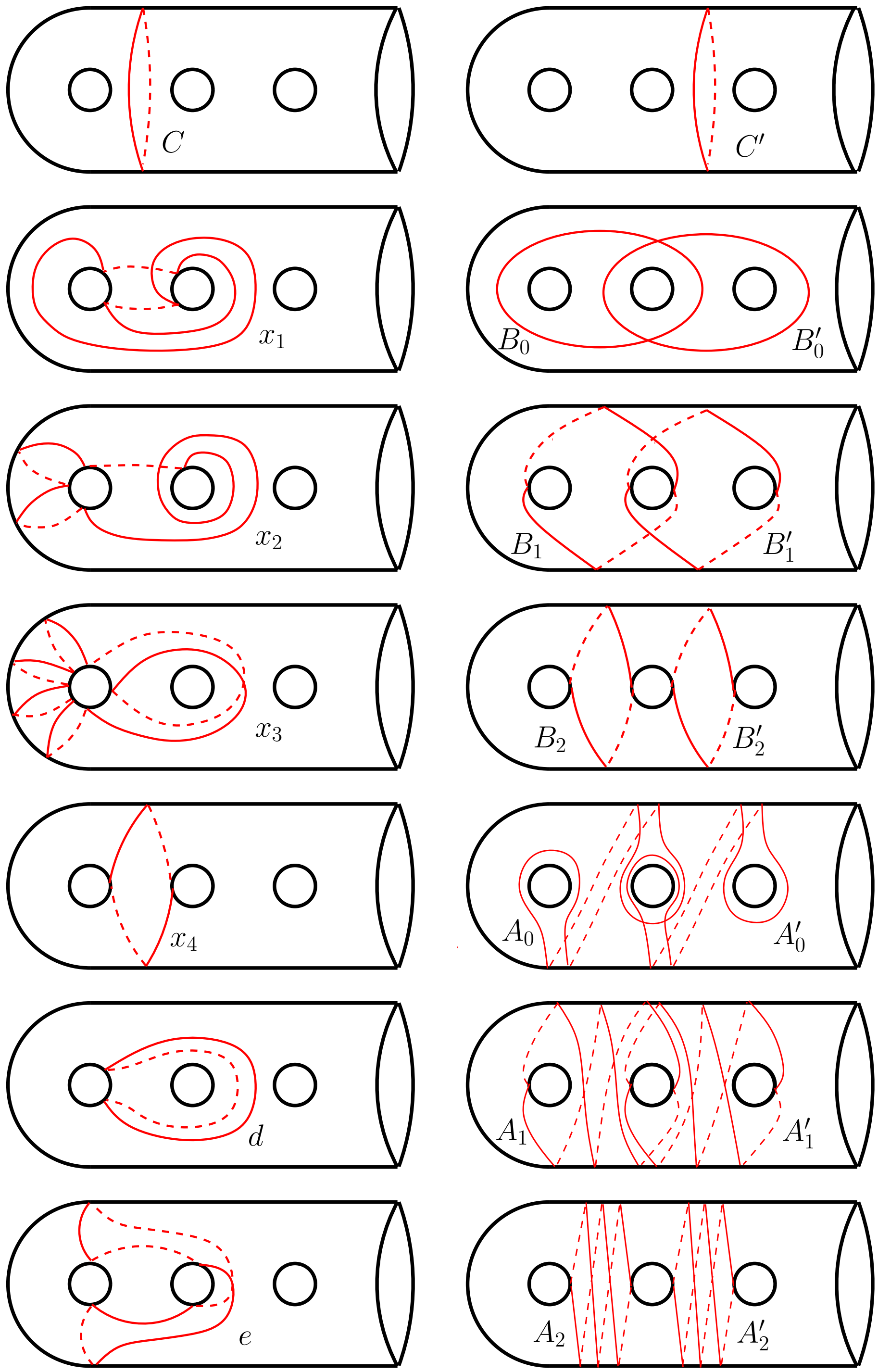

FIGURE 1. The curves C, x_1, x_2, x_3, x_4, d, e of the first embedding are given on the left. The curves B_0, B_1, B_2, A_0, A_1, A_2 of the second embedding (except for C, which is already given on the left) and C', B_0', B_1', B_2', A_0', A_1', A_2' of the third embedding are on the right.

which we rewrite as $P_2' t_{C'}^2 t_C^{-1} = t_\delta$, for $\underline{P_2' = t_{B_0'} t_{B_1'} t_{B_2'} t_{A_0'} t_{A_1'} t_{A_2'}}$. Similarly, P_2', t_C and $t_{C'}$ all commute with each other.

With these three embeddings in hand, we can now describe our positive factorizations. Let ϕ be any mapping class in $\mathrm{Mod}(\Sigma_3^1, S)$, the subgroup of $\mathrm{Mod}(\Sigma_3^1)$ which consists of elements fixing the set $S := \{C, C'\}$ point-wise. Then we have

$$(P_1)^\phi P_1 P_2' t_C = (P_1 t_C t_{C'}^{-1})^\phi \, P_1 t_C t_{C'}^{-1} \, P_2' t_{C'}^2 t_C^{-1} = 1 \cdot t_\partial \cdot 1 = t_\partial \,,$$

where the first equality follows from the commutativity relations noted above and the fact that ϕ commutes with $t_C t_{C'}^{-1}$. The second equality follows from the relations (2)–(4). Therefore $\underline{W_{1,\phi} = (P_1)^\phi P_1 P_2' t_C}$ is a positive factorization of the boundary twist t_∂ in $\mathrm{Mod}(\Sigma_3^1)$. By identical arguments, we see that $\underline{W_{2,\phi} = (P_1)^\phi P_2 P_2' t_C^2}$ and $\underline{W_{3,\phi} = (P_2)^\phi P_2 P_2' t_C^3}$ are also positive factorizations of t_∂ in $\mathrm{Mod}(\Sigma_3^1)$.

Each $W_{i,\phi}$ prescribes a symplectic genus–3 Lefschetz pencil $(X_{i,\phi}, f_{i,\phi})$ with one base point, equipped with a Thurston–Gompf symplectic form. We claim that $\chi_h(X_{i,\phi}) = 1$ and $c_1^2(X_{i,\phi}) = 3 - i$ for each $i = 1, 2, 3$.

The Euler characteristic of $X_{i,\phi}$ is given by

$$\mathrm{e}(X_{i,\phi}) = 4 - 4g + \ell - b = 4 - 4 \cdot 3 + (18 + i) - 1 = 9 + i \,,$$

where g and b are the genus and the number of base points of the pencil, and ℓ is the number of critical points, which is the same as the number of Dehn twists in the positive factorization $W_{i,\phi}$.

Since we have explicit positive factorizations for the pencils $(X_{i,\phi}, f_{i,\phi})$, the signature of each $X_{i,\phi}$ can be easily calculated using the work of Endo–Nagami in [25], which states that the signature of the pencil is equal to the algebraic sum of the signatures of the mapping class group relations used to derive this positive factorization from the trivial word in $\mathrm{Mod}(\Sigma_3^1)$. Since the signature of any embedding of a relation into a higher genus surface is the same, and since Hurwitz moves, conjugations and cancellations of positive–negative Dehn twist pairs do no change the signature, we just need to understand the signatures of the genus–2 relations we used as our building blocks. The signature of the relation (2) is the same as the signature of the genus–2 pencil with one base point on $T^2 \times S^2 \# 2\overline{\mathbb{CP}}^2$, which is -2. The signature of the relation (3) is that of the genus–2 pencil with one base point on $T^2 \times S^2 \# 3\overline{\mathbb{CP}}^2$ (recall that we capped off one of the boundaries first), which is -3. Finally, the signature of the relation (4) is that of the genus–2 pencil with two base points on $T^2 \times S^2 \# 2\overline{\mathbb{CP}}^2$, which is -2. We conclude that $\sigma(X_{i,\phi}) = -5 - i$.

Hence, $\chi_h(X_{i,\phi}) = \frac{1}{4}(\mathrm{e}(X_{i,\phi}) + \sigma(X_{i,\phi})) = \frac{1}{4}(9 + i - 5 - i)) = 1$ for each $i = 1, 2, 3$, whereas $c_1^2(X_{i,\phi}) = 2\,\mathrm{e}(X_{i,\phi}) + 3\,\sigma(X_{i,\phi}) = 2(9 + i) + 3(-5 - i) = 3 - i$, as claimed. Note that the only rational or ruled surfaces which have the same invariants are the rational surfaces $\mathbb{CP}^2 \# \, (6 + i)\overline{\mathbb{CP}}^2$, which by Lemma 2, cannot admit such pencils.

4.2. Breeding pencils with $\chi_h = 1$ and $c_1^2 = 3$.

In our next construction we strive to get *hyperelliptic* pencils. While getting hyperelliptic positive factorizations at every step will constrain some of the freedom we have in our breeding constructions, we will also leverage this additional property whenever we can.

Our construction will be comparable to that of the positive factorization $W_{3,\phi}$ in the previous section, where we employed three embeddings of —various lifts of— the positive factorization for Matsumoto's genus–2 Lefschetz fibration. Here we will get our examples using three different embeddings of —various lifts of— the positive factorization for the genus–2 Lefschetz fibration of Korkmaz and the author in [15]. This positive factorization, after a single Hurwitz move as before, has the following lift in $\mathrm{Mod}(\Sigma_2^3)$:

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_C = t_{\delta_1} t_{\delta_2} t_{\delta_3}\,, \tag{5}$$

where the curves x_i, B_2, C, d, e are as shown in Figure 2. We will simply use the same labels for the Dehn twist curves x_i, B_2, C, d, e for any other relation we derive from (5) by capping off some of the boundary components $\delta_1, \delta_2, \delta_3$.

A comprehensive proof of the relation (5) is given in [12], where it is also shown that this is a positive factorization for a genus–2 pencil with two marked points on the elliptic ruled surface $T^2 \tilde{\times} S^2$. It can also be verified in a straightforward fashion using the Alexander method [26] Below we sketch yet another argument based on the hyperelliptic symmetry of the monodromy curves. This line of arguments can be proved to be useful for similar calculations in general.

Let (X, f) be the hyperelliptic genus–2 Lefschetz fibration corresponding to the positive factorization $t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_C = 1$ in $\mathrm{Mod}(\Sigma_2)$, where $X \cong T^2 \times S^2 \# 3\,\overline{\mathbb{CP}}{}^2$ is equipped with a Thurston–Gompf symplectic form [15]. As shown in [66, 38], there is a symplectic involution on X extending the hyperelliptic involution on the fibers, and f is the relative minimalization of a Lefschetz fibration obtained via the induced symplectic double branched cover $X \# 3\,\overline{\mathbb{CP}}{}^2 \to S^2 \times S^2 \# 6\overline{\mathbb{CP}}{}^2$ (where the blow-ups are for the reducible fibers). The branch set consists of a multisection B of the latter fibration, which intersects every fiber at the fixed points of the hyperelliptic involution, and B_E that consists of exceptional spheres contained in the reducible fibers. Now, observe that when we isotope the monodromy curves of (X, f) so that they are symmetric under the obvious hyperelliptic involution obtained by rotating the surface Σ_2 in Figure 2 by a π–degree rotation along the x–axis (taking z–axis to be perpendicular to the page), they miss the three marked points (drawn in blue in the figure) of the fixed points of the hyperelliptic involution, whereas the four non-separating curves go through the other three points. We can deduce the topology of the branch set from this very data, and in particular conclude that the multisection B consists of three disjoint (-1)–sphere sections E_1, E_2, E_3 (one for each marked point) and a 3–section which is a square zero symplectic 2–sphere (going through the

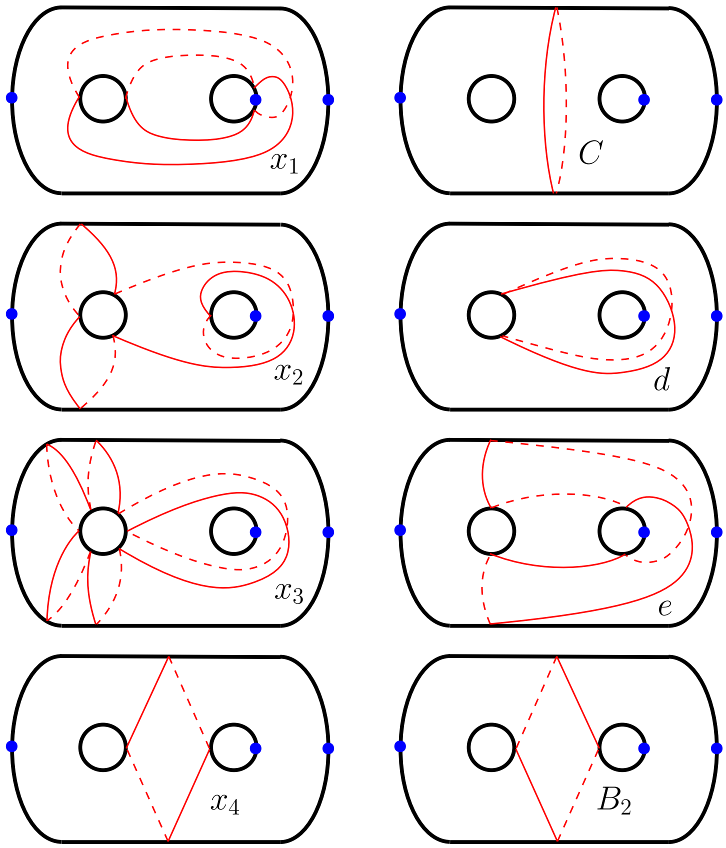

FIGURE 2. The curves C, x_1, x_2, x_3, x_4, d, e in the lift of Baykur–Korkmaz genus–2 positive factorization to $\mathrm{Mod}(\Sigma_2^3)$, along with the curve B_2 one gets after a Hurwitz move

other three fixed points). Circling back to our original discussion, the (-1)–sections E_1, E_2, E_3 yield the lift (5).

We are now ready to describe our three embeddings.

Note that we have now drawn the surface Σ_3^1 so that its boundary curve ∂ is as shown in Figure 3. With this in mind, our first embedding is essentially same as the one yielded the relation (2) in $\mathrm{Mod}(\Sigma_3^1)$: Cap off the boundary components δ_1 and δ_2 of Σ_2^3 and then embed it into Σ_3^1 so that δ_3 maps to C' and the rest of the Dehn twist curves are as shown in Figure 1, except the boundary $\partial\Sigma_3$, which is outside of their support, is shifted. Using the same notation as before, we get the relation $P_1\, t_C t_{C'}^{-1} = 1$ in $\mathrm{Mod}(\Sigma_3^1)$, where $\underline{P_1 = t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2}}$.

For our second embedding, cap off the boundary component δ_3 of Σ_2^3, and then embed the resulting copy of Σ_2^2 into Σ_3^1 so that

$$\delta_1 \mapsto C\,,\; \delta_2 \mapsto \partial\,,\; C \mapsto C'\,,$$

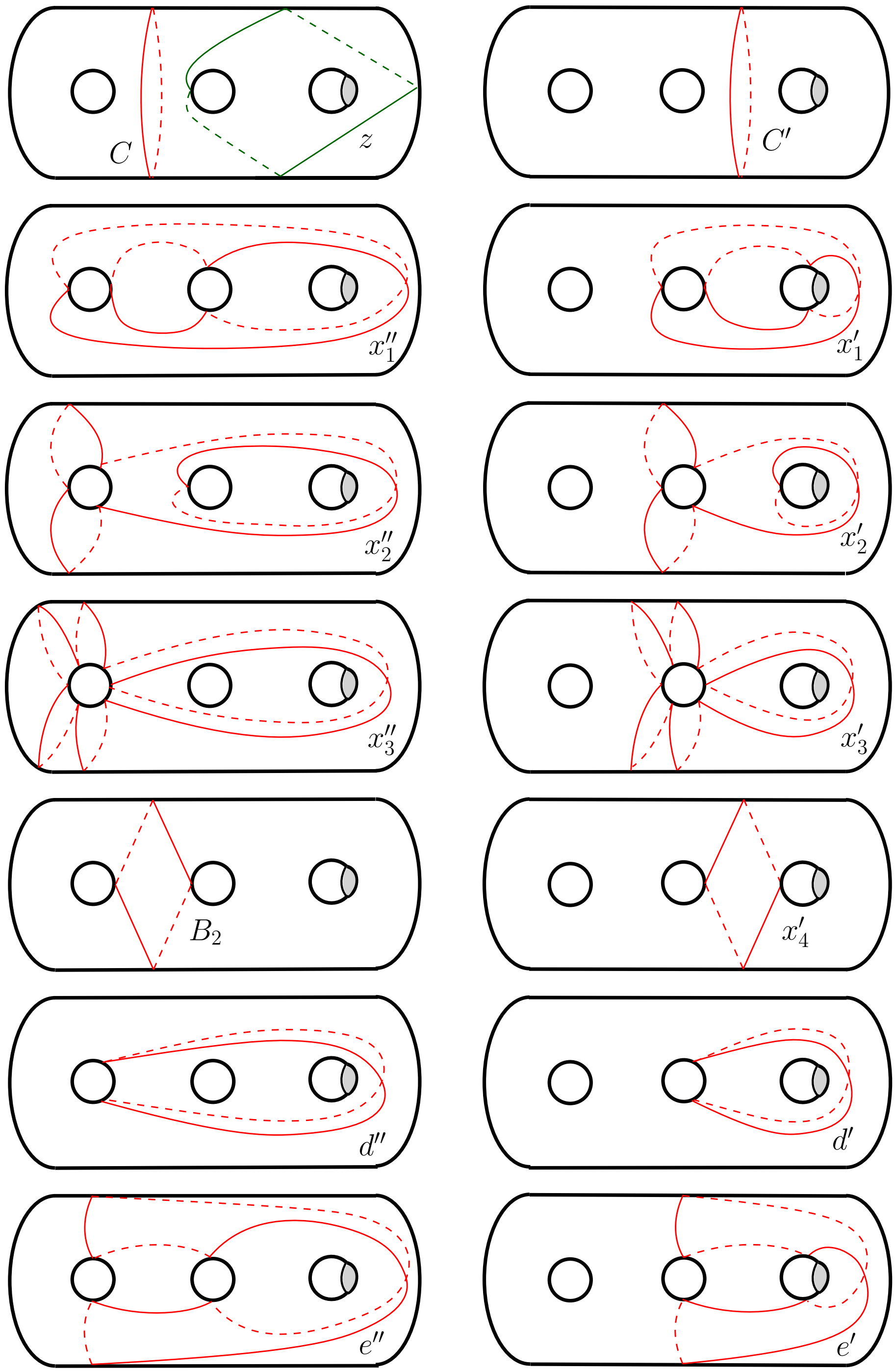

FIGURE 3. The surface Σ_3^1 with shifted boundary. The curves C', x'_1, x'_2, x'_3, x'_4, d', e' of the second embedding are on the right, and the curves C, x''_1, x''_2, x''_3, B_2, d'', e'' of the third embedding are given on the left. The Dehn twist curve z in the conjugation ϕ is on the top left (in green).

and the remaining curves are as shown in Figure 3, where we once again use a prime symbol when denoting the curves by the same letters. For our arguments to follow, here it is more convenient to take the genus–2 relation as $t_{x_4} t_e t_{x_1} t_{x_2} t_{x_3} t_d t_C = t_{\delta_1} t_{\delta_2}$, where we moved t_{B_2} back over t_C by a Hurwitz move and then applied a cyclic permutation. So we get the following relation in $\mathrm{Mod}(\Sigma_3^1)$:

$$t_{x_4'} t_{e'} t_{x_1'} t_{x_2'} t_{x_3'} t_{d'} t_{C'} t_C^{-1} = t_\partial \,, \tag{6}$$

which we rewrite as $P_1'\, t_{d'} t_{C'} t_C^{-1} = t_\delta$, for $\underline{P_1' = t_{x_4'} t_{e'} t_{x_1'} t_{x_2'} t_{x_3'}}$. Importantly, $P_1' t_{d'}$, t_C, $t_{C'}$ and t_∂ all commute with each other.

For our last embedding, cap off the boundary components δ_1 and δ_2 of Σ_2^3, and then embed the resulting copy of Σ_2^1 into Σ_3^1 so that δ_3 maps to the curve d' above, where the curves $e'', x_1'', x_2'', x_3'', d'', B_2, C, d'$ are as shown in Figure 3. (Note that we get the same B_2, C curves.) So we obtain another relation in $\mathrm{Mod}(\Sigma_3^1)$:

$$t_{e''} t_{x_1''} t_{x_2''} t_{x_3''} t_{d''} t_{B_2} t_C t_{d'}^{-1} = 1 \,, \tag{7}$$

which we rewrite as $P_1''\, t_{d'}^{-1} = 1$, for $\underline{P_1'' = t_{e''} t_{x_1''} t_{x_2''} t_{x_3''} t_{d''} t_{B_2} t_C}$. Here P_1'' and $t_{d'}$ commute.

We can now describe our positive factorizations using the three embeddings above. Let ϕ be any mapping class in the stabilizer group $\mathrm{Mod}(\Sigma_3^1, d')$. Then

$$P_1\, P_1'\, (P_1'')^\phi = (P_1\, t_C\, t_{C'}^{-1})\, (P_1'\, t_{d'} t_{C'} t_C^{-1}) (P_1''\, t_{d'}^{-1})^\phi = 1 \cdot t_\partial \cdot 1 = t_\partial \,.$$

Here the first equality follows from the commutativity relations we noted above, along with our choice of ϕ as follows: In the middle, the multi-twist $t_{C'} t_C^{-1}$ commutes with P_1', so we can bring it to its left and cancel it against $t_C t_{C'}^{-1}$. Since ϕ stabilizes the curve d', we have $t_{d'} = (t_{d'})^\phi$, so we can now take the $t_{d'}$ factor into the conjugated expression, and then, because it commutes with P_1'', we can move it to its right and cancel against $t_{d'}^{-1}$ within the parentheses. The second equality is the product of the equalities (2),(6) and (7).

Hence we have obtained a positive factorization $\underline{W_{0,\phi} = P_1 P_1' (P_1'')^\phi}$ of the boundary twist t_δ in $\mathrm{Mod}(\Sigma_3^1)$. Each $W_{0,\phi}$ prescribes a symplectic genus–3 Lefschetz pencil $(X_{0,\phi}, f_{0,\phi})$ with one base point, equipped with a Thurston–Gompf symplectic form.

As before, we can calculate the Euler characteristic of $X_{0,\phi}$ as

$$\mathrm{e}(X_{0,\phi}) = 4 - 4g + \ell - b = 4 - 4 \cdot 3 + 18 - 1 = 9 \,,$$

and the signature of $X_{0,\phi}$ as the signature of the relation $W_{0,\phi}$ after [25]. The latter is equal to the sum of the signatures of the three relations (2),(6) and (7), which correspond to pencils on $T^2 \tilde{\times} S^2 \# 2\overline{\mathbb{CP}}^2$, $T^2 \tilde{\times} S^2 \# \overline{\mathbb{CP}}^2$ and $T^2 \tilde{\times} S^2 \# 2\overline{\mathbb{CP}}^2$, respectively. (In the case of the second embedding, since its second boundary twist t_{δ_2} was mapped to t_∂, it now corresponds to the base point of the genus–3 pencil.) So we get $\sigma(X_{0,\phi}) = -2 - 1 - 2 = -5$. Therefore, $\chi_h(X_{0,\phi}) = 1$ and $c_1^2(X_{0,\phi}) = 3$, as claimed. The only rational or ruled surface which has the same invariants is $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$, which by Lemma 2, cannot admit such pencils.

Lastly, observe that the Dehn twist curves in all three factors P_1, P_1' and P_1'' involved in $W_{0,\phi}$ commute with the obvious involution on Σ_3^1 given by a π–rotation of surface along the x–axis in Figure 3 (taking z–axis perpendicular to the page). If we let $\mathrm{HMod}(\Sigma_3^1)$ denote the symmetric mapping class group with respect to this involution [26], then for any ϕ in the subgroup $\mathrm{Mod}(\Sigma_3^1, d') \cap \mathrm{HMod}(\Sigma_3^1)$, we get a positive factorization $W_{0,\phi}$ prescribing a *hyperelliptic pencil* $(X_{0,\phi}, f_{0,\phi})$.

4.3. Homeomorphism and homology types of $c_1^2 = 0, 1, 2$ examples.

Recall that we have $\mathrm{e}(X_{i,\phi}) = 9 + i$ and $\sigma(X_{i,\phi}) = -5 - i$, for $i = 0, 1, 2, 3$. None of our examples have even intersection forms, which can be easily seen by the existence of reducible fibers in $X_{i,\phi}$, which have self-intersection -1. To be able to pin down the homeomorphism and integral homology types of these 4–manifolds, it remains to determine their fundamental groups and the first integral homology groups, which we will do so for particular choices of ϕ.

Below, we will first carry out these calculations for our examples $(X_{i,\phi}, f_{i,\phi})$ with $i = 1, 2, 3$, and then do the same for the $i = 0$ case in the next subsection. Here we aspire to keep our calculations simple but also generate as many fundamental groups as possible. Finding the right balance will come at a cost of getting a somewhat asymmetric picture; the fundamental groups of $X_{i,\phi}$ will realize any quotient of $\mathbb{Z}^2$ when $i = 1, 3$, and any quotient of $\mathbb{Z}$ when $i = 0, 2$.

Let $\phi = t_{b_1}^{-m_1} t_{a_2}^{m_2}$, where b_1, a_2 are as in Figure 4. Since b_1 and a_2 are disjoint from C and C', we have $\phi \in \mathrm{Mod}(\Sigma_3^1, S)$ for $S = \{C, C'\}$, as required. Denote the positive factorizations in this case by $W_{i,m} := W_{i,\phi}$, where for $i = 1, 3$, we take $\phi = t_{b_1}^{-m_1} t_{a_2}^{m_2}$ and $m = (m_1, m_2) \in \mathbb{N}^2$, whereas for $i = 2$, we take $\phi = t_{b_1}^{-5} t_{a_2}^{m}$ and $m \in \mathbb{N}$. Now, set $(X_{i,m}, f_{i,m}) := (X_{i,\phi}, f_{i,\phi})$, and further set $(X_i, f_i) := (X_{i,m}, f_{i.m})$ in the specific cases of $m = (1, 1)$ when $i = 1, 3$, and $m = 1$ when $i = 2$.

We claim that $\pi_1(X_{i,m})$ is $(\mathbb{Z}/m_1\mathbb{Z}) \oplus (\mathbb{Z}/m_2\mathbb{Z})$, for $i = 1, 3$, and $\mathbb{Z}/m\mathbb{Z}$, for $i = 2$. In particular each X_i is simply–connected.

Let $(\widetilde{X}_{i,m}, \widetilde{f}_{i,m})$ be the Lefschetz fibration we obtain by blowing-up the base points of the pencil $(X_{i,m}, f_{i,m})$. Let $\{a_j, b_j\}$ be the standard generators of $\pi_1(\Sigma_g)$ as shown in Figure 4. Using the standard handlebody decomposition for a Lefschetz fibration with a section, we obtain a finite presentation for $\pi_1(X_{i,m}) = \pi_1(\widetilde{X}_{i,m})$ of the form

$$\langle\, a_1, b_1, a_2, b_2, a_3, b_3 \mid [a_1, b_1][a_2, b_2][a_3, b_3],\ R_{i,m,1}, \ldots, R_{i,m,18+i} \,\rangle\,, \tag{8}$$

where $\{R_{i,m,k}\}_{k=1}^{18+i}$ are relators obtained by expressing the Dehn twist curves in the positive factorization $W_{i,m}$ in the basis $\{a_j, b_j\}_{j=1}^{3}$. We denote the inverse of any fundamental group element x by $\bar{x}$.

In fact, we will first show that a subset of these relators, which come from Dehn twist curves that are all present in *each* factorization $W_{i,m}$, for $i = 1, 2, 3$, already yield an abelian quotient. Since any further quotient will also be abelian, at that point it will suffice to consider only the abelianizations of all the relators $\{R_{i,m,k}\}_{k=1}^{18+i}$.

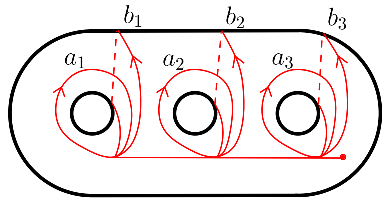

FIGURE 4. Generators a_j, b_j of $\pi_1(\Sigma_3)$

Each positive factorization $W_{i,m}$ contains the factor $P_2' t_C$. So the following relations hold for the finite presentations we have for each $\pi_1(X_{i,m})$:

$$[a_1, b_1][a_2, b_2][a_3, b_3] = 1, \tag{9}$$

$$[a_1, b_1] = 1, \tag{10}$$

$$a_2 a_3 = 1, \tag{11}$$

$$a_2 \bar{b}_2 a_3 \bar{b}_3 = 1, \tag{12}$$

$$b_3 b_2 = 1, \tag{13}$$

where the relators (10)–(13) come from the vanishing cycles C, B_0', B_1', B_2', respectively. We have $\underline{a_3 = \bar{a}_2}$ from (11) and $\underline{b_3 = \bar{b}_2}$ from (13). Together with (12), these imply $\underline{[a_2, b_2] = 1}$ and $[a_3, b_3] = 1$. We conclude $[a_j, b_j] = 1$ for every $j = 1, 2, 3$.

From the factor P_1, we get the following relators (among many others)

$$a_1 (\bar{b}_1 a_2 b_2)^2 = 1, \tag{14}$$

$$a_1 \bar{b}_1^3 a_2 b_2 a_2 = 1, \tag{15}$$

$$a_1 \bar{b}_1^5 a_2 [b_2, a_2] b_1 a_2 = 1, \tag{16}$$

$$b_2 b_1 [b_3, a_3] = 1, \tag{17}$$

induced by the vanishing cycles x_1, x_2, x_3 and B_2, respectively. Adding these to the previous relators from the factor P_2', we immediately see that the commutativity of a_3 and b_3 and (17) imply $\underline{b_1 = \bar{b}_2}$. So (14) implies that $a_1 = (\bar{b}_2 \bar{a}_2 \bar{b}_2)^2$, and since a_2 and b_2 commute, we get $\underline{a_1 = \bar{a}_2^2 \bar{b}_2^4}$

On the other hand, if we have the factor P_2 instead, we get the following relators (again, among many others)

$$a_1 a_2 = 1, \tag{18}$$

$$b_2 \bar{a}_2 b_1 \bar{a}_1 [b_3, a_3] = 1, \tag{19}$$

$$b_2 b_1 [b_3, a_3] = 1,$$

induced by the vanishing cycles B_0, B_1 and B_2, respectively. We get $\underline{a_1 = \bar{a}_2}$, and together with the relators from P_2' we once again get $\underline{b_1 = \bar{b}_2}$, since $[a_3, b_3] = 1$.

Now, since the positive factorization $W_{1,m}$ contains the factor $P_1 P_2' t_C$ and the positive factorizations $W_{2,m}$ and $W_{3,m}$ both contain the factor $P_2 P_2' t_C$, the above discussion shows that every $\pi_1(X_{i,m})$ is a quotient of an abelian group generated by a_2 and b_2. It therefore remains to look at the abelianizations of the relators coming from the remaining Dehn twist curves, i.e. we can simply look at the homology classes of the vanishing cycles.

Without the conjugated factor, we have the abelianized relations

$$a_3 = -a_2 \text{ and } b_3 = -b_2 = b_1 \quad \text{for all } W_{i,m} \tag{20}$$

and depending on weather W_i contains the factor P_1 or P_2, either

$$a_1 + 2a_2 + 4b_2 = 0 \quad \text{for } W_{1,m}, \text{ or} \tag{21}$$

$$a_1 + a_2 = 0 \quad \text{for } W_{2,m} \text{ and } W_{3,m}, \tag{22}$$

where we used (20) to simplify the relators. These relators amount to all the other generators being obtained from a_2 and b_2.

In fact, *there are no other relations* coming from the non-conjugated factors P_1, P_2 or P_2': This is easy to see by abelianizing the relators (9)–(19), which include all the relators induced by the curves $x_1, x_2, x_3, B_0, B_1, B_2, B_0', B_1', B_2'$. Missing are the relators induced by the separating curves d, e from P_1, the curves A_0, A_1, A_2 from P_1, and the curves A_0', A_1', A_2' from P_2'. The first two are trivial in homology, so they have no contribution to the list of relators we already have. On the other hand, for each $j = 0, 1, 2$, A_j is homologous to B_j, because $[A_j] = [t_C(B_j)] = [B_j] + (C \cdot B_j)[C]$, where C is a separating cycle. Similarly each A_j' is homologous to B_j'. Therefore the abelianized relations they induce are identical to those we already had from B_j, B_j'.[2]

It remains to look at the abelianizations of the relators coming from the conjugated factors P_1^ϕ or P_2^ϕ. When $i = 1, 3$, for $\phi = t_{b_1}^{-m_1} t_{a_2}^{m_2}$, we easily check using the Picard-Lefschetz formula that we get the additional relators:

$$b_1 + m_2 a_2 + b_2 = 0 \quad \text{for } W_{1,m} \text{ and } W_{3,m}, \tag{23}$$

$$a_1 + m_1 b_1 + (2 + 4m_2)a_2 + 4b_2 = 0 \quad \text{for } W_{1,m}, \tag{24}$$

$$a_1 + m_1 b_1 + a_2 = 0 \quad \text{for } W_{3,m}. \tag{25}$$

The relations (20) and (23) imply that $\underline{m_2\, a_2 = 0}$. The remaining relators involved in $W_{1,m}$ or $W_{3,m}$ then easily give $\underline{m_1\, b_2 = 0}$. Hence, for $i = 1, 3$, we have

$$\pi_1(X_{i,m}) = (\mathbb{Z}\,/m_1\,\mathbb{Z}) \oplus (\mathbb{Z}\,/m_2\,\mathbb{Z})\,,$$

as claimed.

On the other hand, when $i = 2$, for $\phi = t_{b_1}^{-5} t_{a_2}^{m}$, we get the following additional relators in $W_{2,m}$:

$$b_1 + m a_2 + b_2 = 0 \tag{26}$$

$$a_1 + 5b_1 + (2 + 4m)a_2 + 4b_2 = 0 \tag{27}$$

[2]For the proof of the simply-connected case, one could skip this whole paragraph, since we would only need to find enough relations to kill the fundamental group.

This time, the relations (20) and (26) imply that $\underline{m\, a_2 = 0}$, but then if we use this identity and substitute $a_1 = -a_2$ and $b_1 = -b_2$ into the relator (27), we get $b_2 = a_2$. Therefore, the m–torsion element a_2 generates the whole group. So we have

$$\pi_1(X_{2,m}) = \mathbb{Z}\,/m\,\mathbb{Z}.$$

In particular, when $i = 1, 3$, we get a trivial group for $(m_1, m_2) = (\pm 1, \pm 1)$, and when $i = 2$, we get a trivial group for $m = 1$. So X_i is simply-connected for each $i = 1, 2, 3$. By Freedman's celebrated work [34], each X_i homeomorphic to $\mathbb{CP}^2 \#\, (6{+}i)\, \overline{\mathbb{CP}}{}^2$, for $i = 1, 2, 3$.[3] However, they are not diffeomorphic by Theorem 3.

4.4. **Homeomorphism and homology types of $c_1^2 = 3$ examples.**

Now we take $\phi = t_{b_1}^{-m-10}\, t_z$, where b_1 and z are as in Figures 4 and 3. Since $\phi \in \mathrm{Mod}(\Sigma_3^1, d') \cap \mathrm{HMod}(\Sigma_3^1)$, for each such ϕ, the positive factorization $W_{0,\phi}$ prescribes a *hyperelliptic* genus–3 pencil $(X_{0,\phi}, f_{0,\phi})$. To sync up our notation with the $c_1^2 = 0, 1, 2$ examples, set $W_{0,m} := W_{0,\phi}$ and $(X_{0,m}, f_{0,m}) := (X_{0\phi}, f_{0,\phi})$, while noting that the parameter m takes values in $\mathbb{N}$ (rather than in $\mathbb{N}^2$). Finally, let $(X_0, f_0) := (X_{0,1}, f_{0,1})$. We claim that $\pi_1(X_{0,m}) = \mathbb{Z}\,/\,m\mathbb{Z}$, and in particular X_0 is simply-connected.

As before, we calculate the fundamental group using the presentation of the form (8) induced by the pencil structure. We will first write down only some of the relators we get from the Dehn twist curves in the positive factorization $W_{0,m}$ and observe that any $\pi_1(X_{0,m})$ will be a quotient of an abelian group. It will then suffice to look at the abelianized relators induced by the remaining Dehn twist curves, and run the calculation at the level of homology.

The following relations hold in $\pi_1(X_{0,m})$:

$$[a_1, b_1][a_2, b_2][a_3, b_3] = 1. \tag{28}$$
$$a_1(\bar{b}_1 a_2 b_2)^2 = 1, \tag{29}$$
$$a_1 \bar{b}_1^3 a_2 b_2 a_2 = 1, \tag{30}$$
$$[b_1, a_2 b_2 a_1] = 1, \tag{31}$$
$$[a_2, \bar{b}_1 a_2 b_2] = 1, \tag{32}$$
$$a_2(\bar{b}_2 a_3 b_3)^2 = 1, \tag{33}$$
$$a_3 \bar{b}_3 \bar{a}_3 \bar{b}_2 [a_3, b_3] = 1, \tag{34}$$
$$[a_1, b_1] = 1, \tag{35}$$

where the first one is the surface relation, and (29)–(32) are induced by x_1, x_2, e, d coming from the P_1 factor, (33)–(34) by x_1', x_4' from P_1' and (35) by C from $(P_1'')^\phi$.

[3] Homeomorphism types of other $X_{i,m}$ can also be determined using extensions of Freedman's work by Hambleton, Kreck and Teichner for respective fundamental groups; for example by [45] we can see that when $i = 1, 3$, for $m = (p, 1)$, and when $i = 2$, for $m = p$, each $X_{i,m}$ is homeomorphic to $\mathbb{CP}^2 \#(6+i)\overline{\mathbb{CP}}{}^2 \#\, L_p$, where L_p is the spun of the Lens space $L(p, 1)$.

We can rewrite (30) as $a_1\bar{b}_1^2a_2\bar{b}_1a_2b_2 = 1$ using the commutativity relation (32). Setting this relator equal to the relator (29), we get: $a_1\bar{b}_1^2a_2\bar{b}_1a_2b_2 = a_1\bar{b}_1a_2b_2\bar{b}_1a_2b_2$, which, through cancellations, give $\bar{b}_1a_2 = a_2b_2$. Using this last identity, we can rewrite (31) as $[b_1, \bar{b}_1a_2a_1] = 1$. This implies that $[b_1, a_2a_1] = 1$. However, by (35), b_1 commutes with a_1, so we can further conclude that $[b_1, a_2] = 1$. Since a_2 commutes with b_1, we derive from (32) that $[a_2, b_2] = 1$. In turn, using (28) and (35) we conclude that $[a_3, b_3] = 1$ as well.

We are now ready to show that a_3 and b_1 generate the whole group. Since we saw that $\bar{b}_1a_2 = a_2b_2$, the commutativity of a_2 and b_2 implies that $\underline{b_2 = \bar{b}_1}$. Since a_3 and b_3 commutes, (34) gives $b_3 = \bar{b}_2$, which in turn means $\underline{b_3 = b_1}$. Note that this last identity and the commutativity of a_3 and b_3 now shows that $\underline{[a_3, b_1] = 1}$. Now by (33), we have $a_2 = (\bar{b}_3\bar{a}_3b_2)^2$, which implies that $a_2 = (\bar{b}_1\bar{a}_3\bar{b}_1)^2$. After commuting the factors, we can rewrite the last identity as $\underline{a_2 = \bar{a}_3^2\,\bar{b}_1^4}$. Similarly, by (29), we have $a_1 = (\bar{b}_2\bar{a}_2b_1)^2$, which, after substitutions becomes $a_1 = (b_1(b_1^4a_3^2)b_1)^2$, so $\underline{a_1 = b_1^{12}a_3^4}$.

Underlined equalities we obtained above show that a_3 and b_1 generate the whole group and commute with each other. Hence, $\pi_1(X_{0,m})$ is the quotient of an abelian group with two generators. To finish our calculation of $\pi_1(X_{0,m})$, it now suffices to write out the abelianizations of the relators induced by all the Dehn twist curves in the positive factorization $W_{0,m}$. Clearly the separating Dehn twists do not contribute any non-trivial abelianized relators, whereas each quadruple of non-separating Dehn twists coming from the factors P_1, P_1' and $(P_2'')^\phi$, respectively, can be seen to give only two linearly independent abelianized relators. For instance, the curves x_1, x_2, x_3, B_2 in the P_1 factor yield the relators:

$$a_1 - 2b_1 + 2a_2 + 2b_2 = 0 \tag{36}$$

$$a_1 - 3b_1 + 2a_2 + b_2 = 0, \tag{37}$$

$$a_1 - 4b_1 + 2a_2 = 0, \tag{38}$$

$$b_1 + b_2 = 0, \tag{39}$$

where (38) and (39) generate them all. Similarly, the abelianized relators we get from x_1', x_2', x_3', x_4' in the P_1' factor are generated by

$$a_2 - 4b_2 + 2a_3 = 0, \tag{40}$$

$$b_2 + b_3 = 0, \tag{41}$$

and those we get from x_1'', x_2'', x_3'', B_2 in the non-conjugated P_1'' are generated by

$$a_1 - 4b_1 + 2a_2 + 2a_3 = 0,$$
$$b_1 + b_2 = 0\,.$$

By the Picard-Lefschetz formula, conjugating P_1'' with $\phi = t_{b_1}^{-m-10} t_z$ yields the following additional relators:

$$a_1 + (m+6)b_1 - 2b_2 = 0, \tag{42}$$

$$b_1 + a_2 + 2\, b_2 + a_3 = 0\,. \tag{43}$$

Note that, taking an auxiliary orientation on the twisting curve z, here we have $[z] = a_2 + b_2 + a_3$ in homology.

We can replace the two relations (39) and (41) with $\underline{b_2 = -b_1}$ and $\underline{b_3 = b_1}$. Then (40) can be changed to $\underline{a_2 = -4b_1 - 2a_3}$, and in turn (38) can be changed to $\underline{a_1 = 12b_1 + 4a_3}$. As we express all the other generators in terms of a_3 and b_1, (43) becomes $\underline{a_3 = -5\, b_1}$. Finally, expressing all the generators in terms of b_1, the remaining relation (42) now reads as $\underline{m\, b_1 = 0}$. We conclude that

$$\pi_1(X_{0,m}) = \mathbb{Z}\, / \, m\mathbb{Z}$$

as claimed. When $m = \pm 1$, we get a trivial group, so in particular, $X_0 = X_{0,1}$ is simply-connected. Since we have $\mathrm{e}(X_0) = 9$ and $\sigma(X_0) = -5$, by Freedman [34], X_0 is homeomorphic to $\mathbb{CP}^2 \# \, 6\overline{\mathbb{CP}}^2$, but not diffeomorphic to it by Theorem 3.

4.5. **The theorem and ancillary remarks.**

Combining the results of the previous four subsections, we have:

Theorem 7. *$\{(X_{i,\phi}, f_{i,\phi})\}$ are symplectic genus–3 Lefschetz pencils whose total spaces have $\chi_h(X_{i,\phi}) = 1$ and $c_1^2(X_{i,\phi}) = 3 - i$, and they include exotic rational surfaces $\mathbb{CP}^2 \#(6+i)\overline{\mathbb{CP}}^2$ as well as infinitely many symplectic 4–manifolds which are not homotopy equivalent to any complex surface, for each $i = 0, 1, 2, 3$.*

The additional claim regarding the examples which are not homotopy equivalent to any complex surface follows from standard arguments: The family of symplectic 4–manifolds $\{X_{i,\phi}\}$ contains $\{X_{i,m}\}$ we studied in detail, and the fundamental groups of the latter family realize any $(\mathbb{Z}\, / m_1 \mathbb{Z}) \oplus (\mathbb{Z}\, / m_2 \mathbb{Z})$ for $i = 1, 3$, and any $\mathbb{Z}\, / m\, \mathbb{Z}$, for $i = 0, 2$. For $i = 1, 3$, we get infinitely many examples with $b_1(X_{i,m}) = 1$ (and $b^+(X_{i,m}) > 0$) by setting $m_1 = 0$ and varying m_2. These cannot be homotopy equivalent to any complex surface; see e.g. [9][Lemma 2]. For $i = 0, 2$, we have $\kappa(X_{i,m}) = 2$. However, there are only finitely many deformation classes of compact complex surfaces of general type with the same χ_h and c_1^2 invariants [39], so all but finitely many of these $X_{i,m}$ cannot have the homotopy type of a complex surface.

Remark 8. We claim that $X_{i,\phi}$ are either minimal or *at most once* blow-up of a minimal symplectic 4–manifold. This follows from the following more general observation (cf. Remark 4): For any pencil (X, f), where X is not rational or ruled, the collection of all exceptional classes in the corresponding Lefschetz fibration $(\widetilde{X}, \widetilde{f})$ can be represented by disjoint multisections S_j, each one of which intersects the regular fiber F positively. By [65, Theorem 5-12], $\kappa(X) = 2$ and $g \geq 3$ implies that $(\sum S_j)\cdot F \leq 2g-4$, which in turn means X can have at most $2g-4$ exceptional classes. Note that if $X_{3,\phi}$ is not minimal, then $\kappa(X_{3,\phi}) = 2$, like the other $X_{i,\phi}$, for $i = 0,1,2$. Now for every $i = 0,1,2,3$, since the genus–3 pencil $(X_{i,\phi}, f_{i,\phi})$ already has one base point (yielding an exceptional class in the corresponding fibration), there can be at most one more exceptional class, proving our claim.

Remark 9. We suspect that the smallest exotic rational surface one can get via genus–3 pencils has $c_1^2 = 3$ or 4. and our example (X_0, f_0) might very well be optimal. Our subsequent work in [12] shows that one can already get sharper results with pencils of genus $g = 4$ or 5, which is in part due to having room for more base points, since the number of base points $b \leq 2g-4$ per the previous remarks. It is also worth noting that no exotic rational surface admits a *pencil* of genus $g \leq 2$. The total space of any genus $g \leq 1$ pencil is a rational surface, and that of any genus–2 pencil has $\kappa \geq 1$ by Lemma 2 and Proposition 1. Moreover, by [65, Theorem 5-5(iii)], a genus–2 pencil with $\kappa = 1$ should have only one reducible fiber, which is not possible when the total space has Euler characteristic smaller than 14 by [15, Lemmas 4 and 5]. Hence, the smallest fiber genera for pencils on minimal exotic rational surfaces is $g = 3$. In contrast, there exist genus–2 Lefschetz *fibrations* on minimal exotic rational surfaces with $c_1^2 = 0,1,2$, and in fact for no other c_1^2 [15].

Remark 10. By the work of Siebert and Tian [66], the hyperellipticity of our genus–3 pencil (X_0, f_0) implies that the exotic rational surface X_0 with $c_1^2 = 3$ is the blow-down of a symplectic double cover of a rational ruled surface. The exotic rational surfaces we built in [15] with $c_1^2 = 0,1,2$ via hyperelliptic genus–2 Lefschetz fibrations have the same property. In particular, all these exotic rational surfaces admit symplectic involutions.

Remark 11. There are many prior constructions of Kähler surfaces and symplectic 4–manifolds in the homeomorphism classes of the rational surfaces in Theorem 7. The first examples with $c_1^2 = 0$ and 1 were the Dolgachev surfaces and the Barlow surface, as shown by Donaldson [20] and Kotschick [52], respectively, in the late 1980s. The first examples with $c_1^2 = 2$ and 3 were obtained around 2005 via generalized rational blowdowns by J. Park [63] and Stipsicz–Szabó [70], respectively. Infinitely many distinct smooth structures in these homeomorphism classes were constructed using logarithmic transforms, knot surgeries and Luttinger surgeries; see e.g. [29, 36, 71, 31, 1, 2, 3] (all of which are indeed instances of surgeries along tori [17].) However, it remains an open question whether there are two distinct minimal symplectic 4–manifolds homeomorphic but not diffeomorphic to the same rational surface with $c_1^2 < 9$; see [69][Problem 11]. As observed by Stipsicz and Szabó, Seiberg-Witten invariants cannot distinguish these symplectic 4–manifolds [70][Corollary 4.4]. It is thus desirable to have examples with more structure like ours, in hope of addressing this intriguing question.

Remark 12. It follows from the works of Donaldson [21] and Gompf [40] that every finitely presented group is the fundamental group of a symplectic Lefschetz pencil; also see [5, 51, 44] for direct constructions. One can thus define an invariant m_g of finitely presented groups, where for any such G, $m_g(G)$ is the smallest g among all the genus–g pencils with $\pi_1 = G$. Well-known examples of genus $g = 0, 1, 2$ pencils show that for the groups $G = 1, \mathbb{Z}_2$ and $\mathbb{Z}^2$, we have $m_g = 0, 2, 2$, realized by pencils on $\mathbb{CP}^2$, the Enriques surface, and $T^2 \times S^2$, respectively. We conjecture that $m_g(G) = 3$ for *all the other* $G \cong (\mathbb{Z}/m_1\mathbb{Z}) \oplus (\mathbb{Z}/m_2\mathbb{Z})$, which are realized by our genus–3 pencils $(X_{i,m}, f_{i,m})$, when $i = 0, 1, 2$.

5. Symplectic Calabi-Yau surfaces with $b_1 > 0$ via genus–3 pencils

In this section we will give a new construction of an infinite family of symplectic Calabi-Yau surfaces with $b_1 > 0$ in all possible rational homology classes allowed by the rational homology classification of symplectic Calabi-Yaus [53, 8]. These examples will come from our construction of new positive factorizations of boundary multi-twists in $\mathrm{Mod}(\Sigma_3^4)$ corresponding to symplectic genus–3 Lefschetz pencils.

5.1. **Breeding symplectic Calabi-Yau pencils.**

The positive factorization for Matsumoto's genus–2 Lefschetz fibration has the following further lift to $\mathrm{Mod}(\Sigma_2^4)$, which was obtained by Hamada in [42]:

$$t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1}\, t_{B_{0,2}} t_{B_{1,2}} t_{B_{2,2}} t_{C_2} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} \,, \tag{44}$$

where δ_i are boundary parallel curves, and $B_{j,i}, C_i$ are as shown on the right-hand side of Figure 6. This relation will be the main building block in our construction.

After Hurwitz moves, we can rewrite the relation (44) as

$$t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}}\, t_{A_{0,2}} t_{A_{1,2}} t_{A_{2,2}} t_{C_1} t_{C_2} t_{\delta_1}^{-1} t_{\delta_2}^{-1} = t_{\delta_3} t_{\delta_4} \,,$$

where each $A_{j,2} = t_{C_1}(B_{j,2})$ for $j = 0, 1, 2$. Note that if we cap off the two boundary components δ_3 and δ_4, the curves $B_{j,i}$ descend to the curves B_j and C_i to C given on the left-hand side of Figure 6, for each $j = 0, 1, 2$ and $i = 1, 2$.

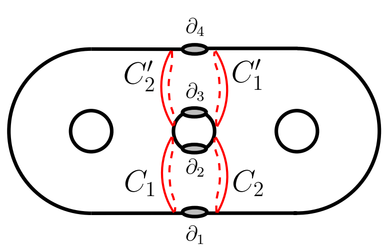

Figure 5. The curves involved in our embeddings of $\partial\Sigma_4^2$ into $\partial\Sigma_3^4$.

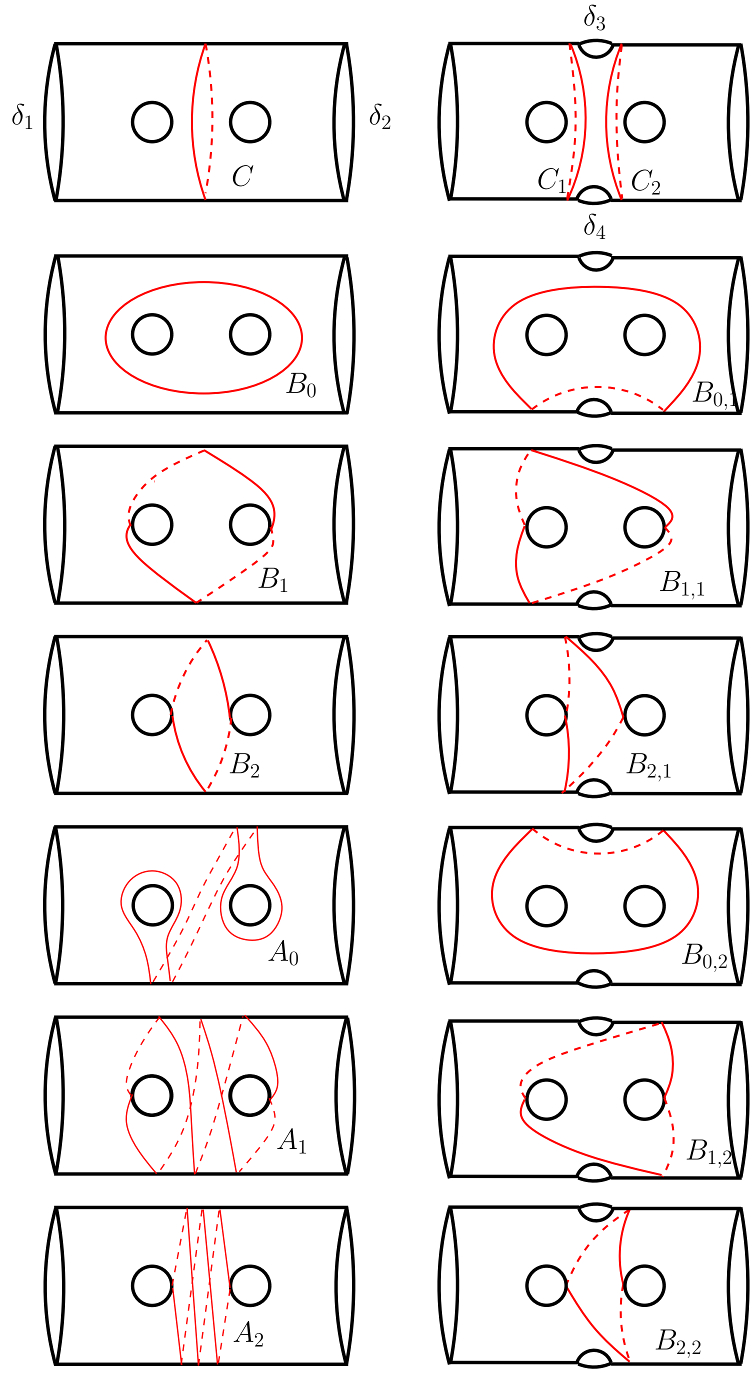

FIGURE 6. The curves $B_j, C, B_{j,i}, C_i$ in Hamada's lifts. On the left are the curves of the positive factorization in $\mathrm{Mod}(\Sigma_2^2)$, along with the curves A_j we got after the Hurwitz moves. On the right are the curves of the further lift in $\mathrm{Mod}(\Sigma_2^4)$.

Consider the embedding of Σ_2^4 into Σ_3^4 obtained by attaching a Σ_0^4 along two of the boundary components of Σ_2^4. We choose an embedding of Σ_2^4 such that we map

$$\delta_1 \mapsto C_2', \, \delta_2 \mapsto C_1', \, \delta_3 \mapsto \partial_2, \, \delta_4 \mapsto \partial_1,$$

where C_i, C_i' are as shown in Figure 5. We then map the interior of Σ_2^4 so that the curves $B_{j,1}$ and $A_{j,2}$ all map to the curves B_j and A_j in Figure 7 when the boundary components $\partial_1, \ldots, \partial_4$ are capped off.[4] Thus, the following relation holds in $\mathrm{Mod}(\Sigma_3^4)$:

$$t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} \, t_{A_{0,2}} t_{A_{1,2}} t_{A_{2,2}} t_{C_1} t_{C_2} t_{C_2'}^{-1} t_{C_1'}^{-1} = t_{\partial_1} t_{\partial_2}, \tag{45}$$

which we can rewrite as $P \, t_{C_1} t_{C_2} t_{C_2'}^{-1} t_{C_1'}^{-1} = t_{\delta_1} t_{\delta_2}$, for $\underline{P = t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} \, t_{A_{0,2}} t_{A_{1,2}} t_{A_{2,2}}}$.

A similar embedding of Σ_2^4 into Σ_3^4 can be given by mapping

$$\delta_1 \mapsto C_2, \, \delta_2 \mapsto C_1, \, \delta_3 \mapsto \partial_3, \, \delta_4 \mapsto \partial_4,$$

where the interior is mapped in a similar fashion as before, so we get the curves B_j' and A_j' in Figure 7 when the boundary components $\partial_1, \ldots, \partial_4$ are capped off. Note that this second embedding can be obtained from the first one by a rotation of the surface Σ_3^4 in Figure 5. So we get another relation in $\mathrm{Mod}(\Sigma_3^4)$:

$$t_{B_{0,1}'} t_{B_{1,1}'} t_{B_{2,1}'} \, t_{A_{0,2}'} t_{A_{1,2}'} t_{A_{2,2}'} t_{C_1'} t_{C_2'} t_{C_2}^{-1} t_{C_1}^{-1} = t_{\partial_3} t_{\partial_4}, \tag{46}$$

which we can rewrite as $P' \, t_{C_1'} t_{C_2'} t_{C_2}^{-1} t_{C_1}^{-1} = t_{\delta_3} t_{\delta_4}$, for $\underline{P' = t_{B_{0,1}'} t_{B_{1,1}'} t_{B_{2,1}'} \, t_{A_{0,2}'} t_{A_{1,2}'} t_{A_{2,2}'}}$.

Now, let ϕ be any mapping class in $\mathrm{Mod}(\Sigma_3^4)$ which fixes the set $S := \{C_1, C_2, C_1', C_2'\}$ point-wise, i.e. $\phi \in \mathrm{Mod}(\Sigma_3^4, S)$. Then the product of P^ϕ and P' yield

$$P^\phi P' = P^\phi \, t_{C_1} t_{C_2} t_{C_2'}^{-1} t_{C_1'}^{-1} \; P' t_{C_1'} t_{C_2'} t_{C_2}^{-1} t_{C_1}^{-1} = (P \, t_{C_1} t_{C_2} t_{C_2'}^{-1} t_{C_1'}^{-1})^\phi \; P' t_{C_1'} t_{C_2'} t_{C_2}^{-1} t_{C_1}^{-1} = \Delta,$$

where $\Delta = t_{\partial_1} t_{\partial_2} t_{\partial_3} t_{\partial_4}$ is the boundary multi-twist. Here, in the first equality we used the commutativity of disjoint Dehn twists t_{C_1}, t_{C_2}, $t_{C_1'}$, $t_{C_2'}$ and that they all commute with P and P'. The second equality holds since ϕ commutes with the Dehn twists along C_1, C_2, C_1' and C_2'.

Therefore $\underline{W_\phi = P^\phi P'}$ is a positive factorization of the boundary multi-twist $\Delta = t_{\partial_1} t_{\partial_2} t_{\partial_3} t_{\partial_4}$ in $\mathrm{Mod}(\Sigma_3^4)$ for any ϕ as above. Under the boundary capping homomorphism $\mathrm{Mod}(\Sigma_3^4) \to \mathrm{Mod}(\Sigma_3)$ this maps to a positive factorization

$$(t_{B_0} t_{B_1} t_{B_2} t_{A_0} t_{A_1} t_{A_2})^\psi \, t_{B_0'} t_{B_1'} t_{B_2'} t_{A_0'} t_{A_1'} t_{A_2'} = 1 \tag{47}$$

where ψ is the image of the mapping class ϕ under this homomorphism.

Let (X_ϕ, f_ϕ) denote the symplectic genus-3 Lefschetz pencil corresponding to the positive factorization W_ϕ. We claim that each X_ϕ is a symplectic Calabi-Yau surface.

[4]At the end of our construction, the four boundary components of Σ_3^4 will correspond to disk neighborhoods of the four base points of our genus–3 pencils, so knowing the isotopy classes of these Dehn twist curves after we cap off all ∂_i will be enough for our π_1 and H_1 calculations.

The Euler characteristic of X_ϕ is easily calculated as

$$\mathrm{e}(X_\phi) = 4 - 4g + \ell - b = 4 - 4 \cdot 3 + 12 - 4 = 0\,,$$

where g and b are the genus and the number of base points of the pencil, and ℓ is the number of critical points, which is equal to the number of Dehn twists in the positive factorization W_ϕ.

As we have an explicit positive factorization (47) for the pencil (X_ϕ, f_ϕ), the signature of X_ϕ can be once again easily calculated using the work of Endo–Nagami in [25]. The signature of the relation (44) we used as our main building block, which corresponds to a pencil on a minimal ruled surface, is zero, and so is the signature of any embedding of this relation into a higher genus surface. Since Hurwitz moves, conjugations and cancellations of positive–negative Dehn twist pairs have no effect on the signature, the signature of the final relation (47) is also zero. Therefore $\sigma(X_\phi) = 0$.

The only rational or ruled surfaces that have the same Euler characteristic and signature as X_ϕ are $T^2 \times S^2$ and $T^2 \tilde{\times} S^2$. However, by Lemma 5 they do not admit pencils with $b = 2g - 2$ base points. Hence, we can apply Proposition 1 to conclude that $\kappa(X_\phi) = 0$. Since X_ϕ clearly does not have the same rational homology as the K3 surface or the Enriques surface, we can already tell that it is a symplectic Calabi-Yau surface with $b_1 > 0$.

5.2. **Homeomorphism and homology types.**

We will first calculate the fundamental group of X_ϕ in the extremal case: when ϕ is the identity and $b_1(X_\phi) = 4$. We will show that the 4–manifold we simply denote by X in this case has $\pi_1(X) = H_1(X) \cong \mathbb{Z}^4$, and we will in fact conclude that X is homeomorphic to the 4–torus. After this detailed calculation, we will calculate $H_1(X_\phi)$ for a certain family of $\phi \in \mathrm{Mod}(\Sigma_3^4, S)$, where $S = \{C_1, C_2, C_1', C_2'\}$, to cover all rational homology types of symplectic Calabi-Yau surfaces with $b_1 > 0$. For any choice of ϕ, one can easily derive a presentation for $\pi_1(X_\phi)$ from that of $\pi_1(X)$, which we will leave to the reader.

Let $(\widetilde{X}, \widetilde{f})$ be the Lefschetz fibration we obtain by blowing-up the base points of the pencil (X, f). Let $\{a_j, b_j\}$ be the standard generators of $\pi_1(\Sigma_g)$ as shown in Figure 4. Once again, we have a finite presentation for $\pi_1(\widetilde{X})$ of the form

$$\langle\, a_1, b_1, a_2, b_2, a_3, b_3 \,|\, [a_1, b_1][a_2, b_2][a_3, b_3],\; R_1, \ldots, R_{12} \,\rangle\,,$$

where each $\{R_k\}_{k=1}^{12}$ is a relation obtained by expressing the Dehn twist curves in the positive factorization (47) in the basis $\{a_j, b_j\}_{j=1}^{3}$.

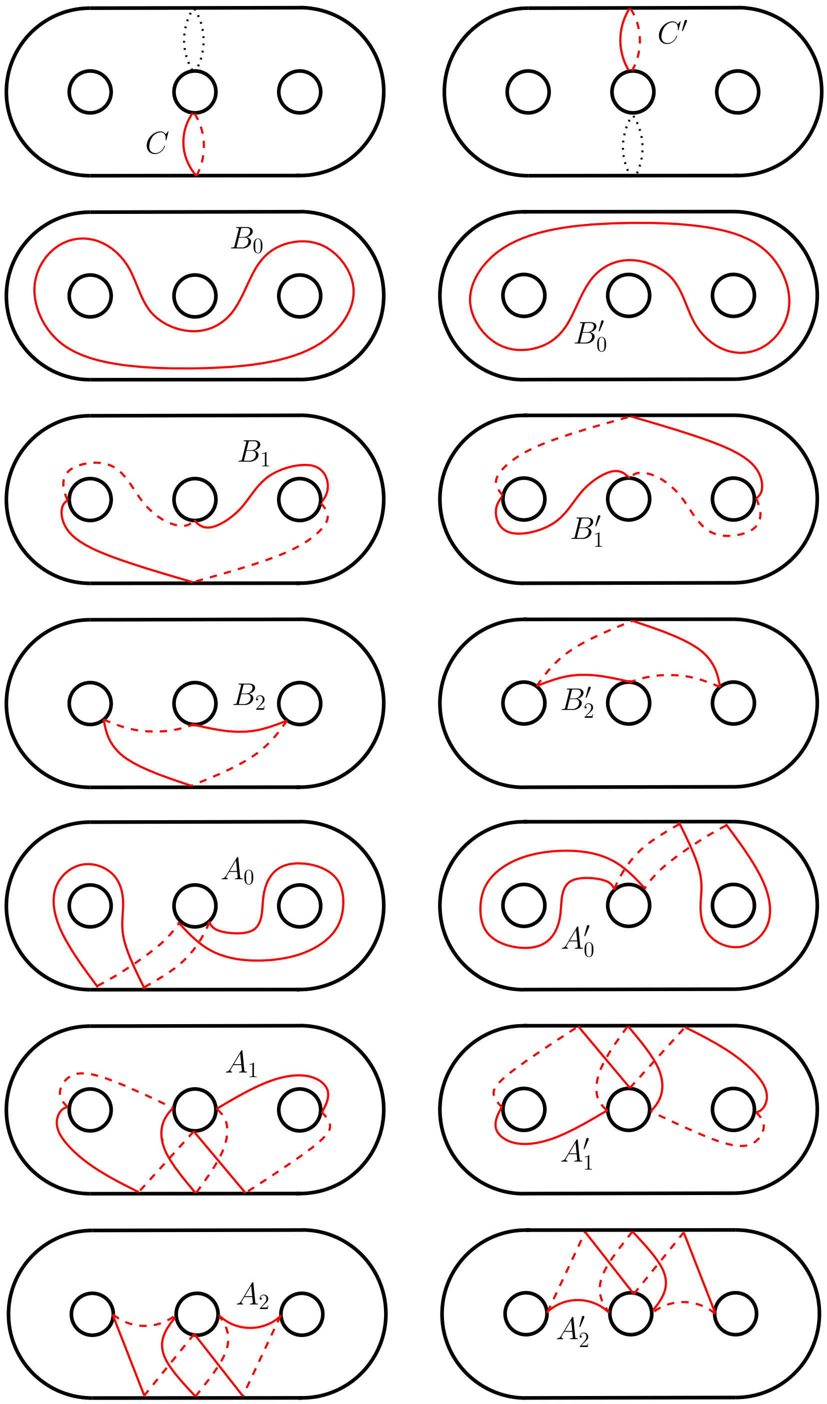

FIGURE 7. The curves B_j, A_j, B'_j, A'_j of the genus–3 pencil (X, f). On the left are the curves coming from the factorization P and on the right are those coming from P', which correspond to the two different embeddings of the factorization in $\mathrm{Mod}(\Sigma_2^2)$ into $\mathrm{Mod}(\Sigma_3)$. (Dotted lines are the identified images of δ_1 and δ_2 under these two embeddings.)

So we have the following relations induced by B_0, B_1, B_2, A_0, A_1, A_2, B_0', B_1', B_2', A_0', A_1', A_2' (see Figure 7), in the same order:

$$a_1 a_3 = 1, \tag{48}$$
$$a_1 \bar{b}_1 a_2 b_2 \bar{a}_2 a_3 \bar{b}_3 = 1, \tag{49}$$
$$\bar{b}_1 a_2 b_2 \bar{a}_2 \bar{b}_3 = 1, \tag{50}$$
$$a_1 [b_3, a_3]\, b_2 a_3 \bar{b}_2\, [a_3, b_3] = 1, \tag{51}$$
$$a_3 \bar{b}_3 \bar{b}_2 [a_3, b_3]\, a_1^2 \bar{b}_1 \bar{a}_1 [b_3, a_3]\, b_2 [b_3, a_3]\, b_2 = 1, \tag{52}$$
$$a_1 \bar{b}_1 \bar{a}_1 [b_3, a_3]\, b_2 [b_3, a_3]\, b_2 \bar{b}_3 \bar{b}_2 [a_3, b_3] = 1, \tag{53}$$
$$\bar{a}_2 a_1 a_2 a_3 = 1, \tag{54}$$
$$a_1 \bar{b}_1 a_2 a_3^2\, \bar{b}_3 \bar{a}_3 b_2 \bar{a}_2 = 1, \tag{55}$$
$$b_1 a_2 \bar{b}_2 a_3 b_3 \bar{a}_3 \bar{a}_2 = 1, \tag{56}$$
$$a_1 a_2 \bar{b}_2 a_3 b_2 \bar{a}_2 = 1, \tag{57}$$
$$a_1 \bar{b}_1 a_2 \bar{b}_2 a_3^2\, \bar{b}_3 \bar{a}_3 b_2^2\, \bar{a}_2 = 1, \tag{58}$$
$$\bar{b}_1 a_2 \bar{b}_2 a_3 \bar{b}_3 \bar{a}_3 b_2^2\, \bar{a}_2 = 1. \tag{59}$$

First observe that, when abelianized, the relations coming from each triple $\{B_0, B_1, B_2\}$, $\{A_0, A_1, A_2\}$, $\{B_0', B_1', B_2'\}$, $\{A_0', A_1', A_2'\}$ yield the same three relations

$$\begin{aligned} &a_1 + a_3 = 0, \\ &a_1 - b_1 + b_2 + a_3 - b_3 = 0, \\ &b_1 - b_2 + b_3 = 0, \end{aligned}$$

where we identified the abelianized images of the π_1 generators with the same letters. Any two of these relations imply the third. Since $a_1 = -a_3$ and $b_1 = b_2 - b_3$, we can eliminate a_1 and b_1 (and these relations) from the presentation, and we get free abelian group of rank 4, generated by a_2, b_2, a_3 and b_3.

Now, going back to the presentation we had for $\pi_1(\widetilde{X})$, we see that it is also generated by a_2, b_2, a_3, b_3, for $a_1 = \bar{a}_3$ by (48) and $b_1 = a_2 b_2 \bar{a}_2 \bar{b}_3$ by (50). Therefore, to conclude that $\pi_1(\widetilde{X}) = \mathbb{Z}^4$, it suffices to show that a_2, b_2, a_3 and b_3 all commute with each other, which is what do next: Replacing a_1 with $\bar{a}_3$ in (54) gives $\underline{[a_2, a_3] = 1}$. From (50) we have $\bar{b}_1 a_2 b_2 \bar{a}_2 = b_3$. Substituting this in (49), and replacing a_1 with $\bar{a}_3$, we get $\underline{[a_3, b_3] = 1}$. With $a_1 = \bar{a}_3$ and $[a_3, b_3] = 1$, the relation (51) simplifies to $\underline{[b_2, a_3] = 1}$. So a_3 commutes with a_2, b_2 and b_3, and therefore, with everything. Since $a_1 = \bar{a}_3$, the surface relation $[a_1, b_1][a_2, b_2][a_3, b_3] = 1$ becomes $\underline{[a_2, b_2] = 1}$. Recall that $b_1 = a_2 b_2 \bar{a}_2 \bar{b}_3$, which now becomes $b_1 = b_2 \bar{b}_3$. Substituting $a_1 = \bar{a}_3$ and $b_1 = b_2 \bar{b}_3$ into (53), and then simplifying it using all the commutativity relations we have so far, we get $\underline{[b_2, b_3] = 1}$. Finally, commuting and canceling the a_1 and a_3 terms in the relation (55) we get $\bar{b}_1 a_2 \bar{b}_3 b_2 \bar{a}_2 = 1$, which, we can rewrite as

$b_3\bar{b}_2a_2\bar{b}_3b_2\bar{a}_2 = 1$ by substituting $b_1 = b_2\bar{b}_3$. Since b_2 commutes with both a_2 and b_3, we can simplify the last relation to get $\underline{[a_2, b_3] = 1}$.

Hence $\pi_1(X) = \pi_1(\widetilde{X}) = \mathbb{Z}^4$, generated by a_2, b_2, a_3 and b_3.

Since $\pi_1(X) = \mathbb{Z}^4$ is a virtually poly-$\mathbb{Z}$ group, the Borel conjecture holds in this case by the work of Farrell and Jones [27]. As observed by Friedl and Vidussi, this implies that a symplectic Calabi Yau surface with $\pi_1 = \mathbb{Z}^4$ is unique up to homeomorphism [35]. So X is homeomorphic to the 4–torus.

Lastly, we will show that for suitable choices of ϕ, we can get X_ϕ realizing all possible rational homology types of symplectic Calabi-Yau surfaces with $b_1 > 0$, which are precisely the rational homology types of torus bundles over tori [55]. In fact, we will get an infinite family realizing all *integral* homology types of torus bundles over tori. Because the Euler characteristic and the signature are fixed (both zero), the first homology groups determine all the others. Therefore, it will suffice to show that we can get X_ϕ with $H_1(X_\phi) = \mathbb{Z}^2 \oplus (\mathbb{Z}/m_1\mathbb{Z}) \oplus (\mathbb{Z}/m_2\mathbb{Z})$ for any given $m_1, m_2 \in \mathbb{N}$.

Let us take $\phi = t_{b_1}^{-m_1} t_{a_3}^{m_2}$, where b_1 and a_3 are as in Figure 4. Note that b_1 and a_3 are disjoint from C_1, C_2, C_1', C_2', so ϕ fixes this set of curves point-wise. For $m = (m_1, m_2)$ any pair of non-negative integers, let us denote the genus–3 pencil we obtain this way by (X_m, f_m) and its positive factorization by $\underline{W_m = P^\phi P'}$, where $\phi = t_{b_1}^{-m_1} t_{a_3}^{m_2}$. Note that $X_{(0,0)} = X$.

Recall that, every triple of vanishing cycles $\{B_0, B_1, B_2\}$, $\{A_0, A_1, A_2\}$, $\{B_0', B_1', B_2'\}$, $\{A_0', A_1', A_2'\}$ yield the two linearly independent relations

$$a_1 + a_3 = 0, \tag{60}$$

$$b_1 - b_2 + b_3 = 0\,. \tag{61}$$

There are two conclusions to draw: First, the Dehn twist curves coming from the non-conjugated factor P' induce exactly these relations in $H_1(X_m)$. Second, the vanishing cycles coming from the conjugated factor P^ϕ induce the following relations, which we can easily derive using the Picard-Lefschetz formula:

$$a_1 + m_1 b_1 + a_3 = 0, \tag{62}$$

$$b_1 - b_2 + m_2 a_3 + b_3 = 0\,. \tag{63}$$

We can now easily see that (60) and (62) together imply $m_1 b_1 = 0$, whereas (61) and (63) imply $m_2 a_3 = 0$. From the relations $a_3 = -a_1$ and $b_3 = b_2 - b_1$, we then conclude that $H_1(X_m)$ is generated by a_1, b_1, a_2, b_2 with only two relations: $m_1 b_1 = 0$ and $m_2 a_1 = 0$. Hence, $H_1(X_m) = \mathbb{Z}^2 \oplus (\mathbb{Z}/m_1\mathbb{Z}) \oplus (\mathbb{Z}/m_2\mathbb{Z})$, as claimed.

Note that when $m_1 = m_2 = \pm 1$, we get symplectic Calabi-Yau surfaces with the same integral homology type as $T^2 \times S^2$, but obviously not diffeomorphic to it, as they have different Kodaira dimensions.

5.3. **The theorem and final remarks.**

For $\{(X_\phi, f_\phi)\}$ symplectic genus–3 pencils prescribed by the positive factorizations W_ϕ, for $\phi \in \mathrm{Mod}(\Sigma_3^4, S)$, we have now proved that:

Theorem 13. $\{(X_\phi, f_\phi)\}$ *are symplectic genus–*3 *Lefschetz pencils whose total spaces are symplectic Calabi-Yau surfaces that realize all integral homology types of torus bundles over tori, and they include a symplectic Calabi-Yau surface homeomorphic to the* 4*–torus and fake symplectic* $T^2 \times S^2$*s.*

We finish with a few observations and comparisons regarding our examples.

Remark 14. The most curious question about our examples is whether every X_ϕ is a torus bundle over a torus, as they are commonly conjectured to exhaust all the diffeomorphism types of symplectic Calabi-Yau surfaces with $b_1 > 0$. After the first version of our paper was publicized, Hamada and Hayano succeeded to prove that our symplectic Calabi-Yau surface that is homeomorphic to the 4–torus [43], is in fact diffeomorphic to it, by comparing the pencil we described on it with a pencil described by Ivan Smith on the standard 4–torus [68] (more on this below). This is so far the only example we know to be standard within this infinite family of examples.[5] If for any conjugation $\phi \neq 1$, it turns out that $\pi_1(X_\phi)$ is not a 4–dimensional solvmanifold group [48, 35], this would imply that X_ϕ is *not* a torus bundle over a torus, and is a new symplectic Calabi-Yau surface. As our arguments in the proof of Theorem 13 show, more generally, if any partial conjugation along any Hurwitz equivalent factorization to the positive factorization W_ϕ results in a pencil with a fundamental group which is not a solvmanifold group, we can arrive at a similar conclusion. So far, a handful of examples we examined seem to have the same group theoretic properties as their infrasolvmanifold counter-parts; e.g. they are poly-$\mathbb{Z}$ of Hirsch length 4. In particular, we don't know at this point if any of the fake symplectic $T^2 \times S^2$ we get has $\pi_1 = \mathbb{Z}^2$ so that it would be exotic.

Remark 15. In [68], Ivan Smith constructed genus–3 pencils on torus bundles over tori admitting sections (not all do), by generalizing the algebraic geometric construction of holomorphic genus–3 pencils on abelian surfaces. It is natural to ask whether our examples overlap with Smith's. As Hamada and Hayano showed in [43], this is the case for our 4–torus example, but we don't know much about it beyond that. There are however reasons to think that our family of genus–3 pencils (X_ϕ, f_ϕ) is at least larger than Smith's examples. For comparison, note that any torus bundle over a torus with a section would admit a second disjoint section as well; for any section of a surface bundle over a torus has self-intersection zero [16] and can be pushed-off of itself. So the family of genus–3 pencils of Smith are

[5]It might be possible to use the recent works of W. Chen in [18, 19] to conclude that some other X_ϕ are also standard by finding finite symplectic symmetries on them. In the special case of trivial ϕ, one can in fact see that the monodromy of the pencil (X, f) with $b_1(X) = 4$ has a $\mathbb{Z}_2$–symmetry under cyclic permutation, which gives rise to a symplectic involution on X.

determined by a pair $\phi_1, \phi_2 \in \mathrm{Mod}(\Sigma_1^2)$ *subject to the relation* $[\phi_1, \phi_2] = 1$. On the other hand, our family of genus–3 pencils are parameterized by $\phi \in \mathrm{Mod}(\Sigma_3^4, S)\}$, where $S := \{C_1, C_2, C_1', C_2'\}$, which has a proper subgroup that consists of mapping classes which fix each one of the curves C_1, C_2, C_1', C_2'. Any ϕ in the latter stabilizer group has disjoint support in two copies of Σ_1^2 embedded in Σ_3^4 (the left and the right sides of the surface in Figure 5). So a subset of our family of examples are also parameterized by $\phi_1, \phi_2 \in \mathrm{Mod}(\Sigma_1^2)$, *but with no relation to each other whatsoever.*

Remark 16. The subfamily of pencils $\{(X_m, f_m) \,|\, m = (m_1, m_2) \in \mathbb{N}^2\}$ we studied in the proof of Theorem 13 have the following property: they can all be obtained from the 4–torus pencil (X, f) through *fibered Luttinger surgeries* [6, 10]. To see this, first observe that for $\phi = t_{b_1}^{-m_1} t_{a_3}^{m_2}$, we have the positive factorizations

$$W_m = (t_{b_1}^{-m_1} t_{a_3}^{m_2} \, P \, t_{a_3}^{-m_2} t_{b_1}^{m_1}) \, P' = (t_{b_1}^{-1} \cdots t_{b_1}^{-1} t_{a_3} \cdots t_{a_3} \, P \, t_{a_3}^{-1} \cdots t_{a_3}^{-1} t_{b_1} \cdots) t_{b_1} \, P' \, .$$

which are obtained by a *sequence* of partial conjugations by t_{b_1} and t_{a_3}. Since b_1 and a_3 are disjoint from C_1, C_1', C_2, C_2', they are stabilized by P, which, as a mapping class, equals to $t_{C_1}^{-1} t_{C_2}^{-1} t_{C_1'} t_{C_2'}$. So each conjugation by a factor of $t_{b_1}^{\pm 1}$ or $t_{a_3}^{\pm 1}$ amounts to performing a Luttinger surgery along a Lagrangian torus swept off by b_1 or a_3 on the regular fibers, over a loop on the base [6, 10]. One can easily see how this observation generalizes to more general conjugations (but perhaps requiring Luttinger surgeries along Lagrangian Klein bottles). With this in mind, we see that if $\{(X_\phi, f_\phi)\}$ contains all the torus bundles over tori, then one would immediately get a proof of an improved version of a conjecture by C.-I. Ho and T.-J. Li: that every torus bundle over a torus admits a symplectic structure so that it is obtained via Luttinger surgeries along tori from the 4–torus equipped with the standard product symplectic structure [47, Conjecture 4.9], or Klein bottles, we add.

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