Families of 4-Manifolds with Nontrivial Stable Cohomotopy Seiberg-Witten Invariants, and Normalized Ricci Flow

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Abstract

In this article, we produce infinite families of 4-manifolds with positive first betti numbers and meeting certain conditions on their homotopy and smooth types so as to conclude the non-vanishing of the stable cohomotopy Seiberg-Witten invariants of their connected sums. Elementary building blocks used in [28] are shown to be included in our general construction scheme as well. We then use these families to construct the first examples of families of closed smooth 4-manifolds for which Gromov's simplicial volume is nontrivial, Perelman's $\bar{\lambda}$ invariant is negative, and the relevant Gromov-Hitchin-Thorpe type inequality is satisfied, yet no non-singular solution to the normalized Ricci flow for any initial metric can be obtained. In [12], Fang, Zhang and Zhang conjectured that the existence of any non-singular solution to the normalized Ricci flow on smooth 4-manifolds with non-trivial Gromov's simplicial volume and negative Perelman's $\bar{\lambda}$ invariant implies the Gromov-Hitchin-Thorpe type inequality. Our results in particular imply that the converse of this fails to be true for vast families of 4-manifolds.

1 Introduction

Let X be a closed smooth Riemannian 4-manifold X with $b^+(X) > 1$, where $b^+(X)$ denotes the dimension of the maximal positive definite linear subspace in the second cohomology of X. In what follows, $e(X)$ and $\text{sign}(X)$ denote respectively the Euler characteristic and signature of X. Recall that a spin^c-structure Γ_X on X induces a pair of spinor bundles $S^{\pm}_{\Gamma_X}$ which are Hermitian vector bundles of rank 2 over X. A Riemannian metric on X and a unitary connection A on the determinant line bundle $\mathcal{L}_{\Gamma_X} := \det(S^+_{\Gamma_X})$ induce the twisted Dirac operator $\mathcal{D}_A : \Gamma(S^+_{\Gamma_X}) \longrightarrow \Gamma(S^-_{\Gamma_X})$. The Seiberg-Witten monopole equations [48] over X are the following non-linear partial differential equations for a unitary connection A of the complex line bundle $\mathcal{L}_{\Gamma_X}$ and a spinor $\phi \in \Gamma(S^+_{\Gamma_X})$:

$$\mathcal{D}_A\phi = 0, \ F^+_A = iq(\phi),$$

here F^+_A is the self-dual part of the curvature of A and $q : S^+_{\Gamma_X} \to \wedge^+$ is a certain natural real-quadratic map, where $\wedge^+$ is the bundle of self-dual 2-forms. The quotient space of the set of solutions to the Seiberg-Witten monopole equations by gauge group is called the Seiberg-Witten moduli space. In his celebrated article [48], Witten introduced an invariant of smooth 4-manifolds by using the fundamental homology class of the Seiberg-Witten moduli space, which is now called the *Seiberg-Witten invariant*, and is well-defined for any closed 4-manifold X with $b^+(X) > 1$.

In [7, 5], Bauer and Furuta adopted a remarkable approach to introduce a refinement of the Seiberg-Witten invariant SW_X without using the Seiberg-Witten moduli space. They introduced a new invariant, which takes values in a certain stable cohomotopy group $\pi^{b^+}_{S^1,\mathcal{B}}(\mathrm{Pic}^0(X), \mathrm{ind}D)$, where $b^+ := b^+(X)$ and $\mathrm{ind}D$ is the virtual index bundle for the Dirac operators parametrized by the $b_1(X)$-dimensional Picard torus $\mathrm{Pic}^0(X)$. This invariant is called the *stable cohomotopy Seiberg–Wittten invariant*, and herein will be denoted as:

$$BF_X(\Gamma_X) \in \pi^{b^+}_{S^1,\mathcal{B}}(\mathrm{Pic}^0(X), \mathrm{ind}D).$$

where Γ_X is a spin^c-structure on X. Moreover, in [5] Bauer proved a non-vanishing theorem of BF_* for a connected sum of 4-manifolds with $b^+ > 1$ and $b_1 = 0$ [5] subject to a couple of conditions, and used this theorem to show that there are 4-manifolds that appear as such connected sums, for which SW_* is trivial but BF_* is not. In particular, BF_* is a strictly stronger invariant than SW_*.

In [28], H. Sasahira and the second author of the current article generalized Bauer's non-vanishing theorem by removing the condition $b_1 = 0$ for the summands. We now introduce the notion of *BF-admissibility* for a 4-manifold, as discussed in [28]:

Definition 1 *A closed oriented smooth* 4*-manifold* X *with* $b^+(X) > 1$ *is called BF-admissible if the following three conditions are satisfied.*

1. *There exists a spin*^{c}*-structure* Γ_X *with* $SW_X(\Gamma_X) \equiv 1 \pmod 2$ *and* $c_1^2(\mathcal{L}_{\Gamma_X}) = 2e(X) + 3sign(X)$*, where* $c_1(\mathcal{L}_{\Gamma_X})$ *is the first Chern class of* $\mathcal{L}_{\Gamma_X}$*.*

2. $b^+(X) - b_1(X) \equiv 3 \pmod 4$.

3. $\mathfrak{S}^{ij}(\Gamma_X) := \frac{1}{2} < c_1(L_{\Gamma_X}) \cup \mathfrak{e}_i \cup \mathfrak{e}_j, [X] > \equiv 0 \pmod 2$ *for all i, j,*

where $\mathfrak{e}_1, \mathfrak{e}_2, \cdots, \mathfrak{e}_s$ *be a set of generators of* $H^1(X, \mathbb{Z})$*,* $s = b_1(X)$*, and* $[X]$ *is the fundamental class of* X_i *and* $< \cdot, \cdot >$ *is the pairing between cohomology and homology.*

The new non-vanishing theorem [28] tells us that, for $i = 1, 2, 3$, a connected sum $\#_{i=1}^{j} X_i$ of BF-admissible, closed oriented 4-manifold X_i has a non-trivial stable cohomotopy Seiberg-Witten invariant. Observe that, when $b_1(X_i) = 0$, the second condition for BF-admissibility just reads as $b^+(X) \equiv 3 \pmod 4$ and the third one holds trivially. That is, the new non-vanishing theorem when $b_1(X_i) = 0$ for all summands is nothing but Bauer's non-vanishing theorem from [5], and therefore can be regarded as a natural generalization of the latter. In order to apply this new non-vanishing theorem of stable cohomotopy Seiberg-Witten invariant to geometry and topology of smooth 4-manifolds, it is essential to find BF-admissible 4-manifolds. Of particular interest was to find BF-admissible 4-manifolds with $b_1 \neq 0$, so as to get new applications that does not follow from Bauer's original non-vanishing theorem stated for $b_1 = 0$. In [28], two types of 4-manifolds were seen to be BF-admissible: Products $\Sigma_g \times \Sigma_h$ of two Riemann surfaces of odd genera, and primary Kodaira surfaces. Failing to get other examples of 4-manifolds with $b_1 > 0$ satisfying the BF-axioms, the authors raised the following problem in the same work [28]:

Problem 1 ([28]) *Find BF-admissible, closed oriented* 4*-manifolds with* $b_1 > 0$*, which are not primary Kodaira surfaces or products* $\Sigma_g \times \Sigma_h$ *of Riemann surfaces with odd genera.*

In the first part of our article, we will answer this problem by showing the existence of vast families of BF-admissible 4-manifolds with $b_1 > 0$. Moreover, we will see that these families naturally include products $\Sigma_g \times \Sigma_h$ and primary Kodaira surfaces. The main surgical operation involved in these constructions is the *Luttinger surgery* along Lagrangian tori [37], defined and discussed in detail in Subsection 2.1 below.

In Subsection 2.2, we will introduce the notion of *surgered product manifolds* which are obtained from products $\Sigma_g \times \Sigma_h$ via Luttinger surgeries along certain homologically essential Lagrangian tori. Note that $\Sigma_g \times \Sigma_h$ are the trivial examples of surgered product manifolds. We will prove that:

Theorem A *Let* $\Sigma_g \times \Sigma_h$ *be the product of two Riemann surfaces of odd genera* g, h*, equipped with the product symplectic form. Then any surgered product manifold obtained from* $\Sigma_g \times \Sigma_h$ *with* $b_1 > 0$ *is BF-admissible. Moreover, and primary Kodaira surface is a surgered product manifold obtained from* $T^2 \times T^2$*, and is BF-admissible.*

In [1], Akhmedov, Baldridge, Kirk, D. Park, and the first author of the current article, showed that a very large portion of the symplectic geography plane could be populated with minimal symplectic 4-manifolds. In Subsection 2.3, we will make use of these examples, while paying attention to preserving BF-admissibility during the employed surgical operations, to prove the following:

Theorem B *Let* a *and* b *be integers satisfying* $2a+3b \geq 0$, $a+b \equiv 0 \pmod 8$*, and* $b < -1$ *is satisfied. Set as* $\alpha = (a+b)/2$ *and* $\beta = (a-b)/2$*. Then, there exists a BF-admissible, irreducible symplectic* 4*-manifold with fundamental group* $\mathbb{Z}$ *which is homeomorphic to*

$$\alpha\mathbb{C}P^2 \# \beta\overline{\mathbb{C}P^2} \# (S^1 \times S^3) \tag{1}$$

and a BF-admissible, irreducible symplectic 4*-manifold with fundamental group* $\mathbb{Z}_p$*,* p *odd, which is homeomorphic to*

$$(\alpha - 1)\mathbb{C}P^2 \# (\beta - 1)\overline{\mathbb{C}P^2} \# Y_p, \tag{2}$$

where Y_p *is the* 4*-manifold with fundamental group* $\mathbb{Z}_p$*, obtained from the product* $L(p,1) \times S^1$ *of Lens space* $L(p,1)$ *and* S^1 *after a* 0 *surgery along* $\{pt\} \times S^1$*.*

Note that these symplectic 4-manifolds are not brand new; they are produced using the families of [1], and were studied in [46]. The new key observation is that, under the mild condition $a + b \equiv 0 \pmod 8$, they are all BF-admissible.

Combining the new non-vanishing theorem [28], Theorems A, and B, we conclude that vast families that consist of connected sums of 4-manifolds with $b_1 > 0$ have non-trivial stable cohomotopy Seiberg-Witten invariants. The existence of such families of connected sums enables us to give several new application regarding the geometry and topology of smooth 4-manifolds, which we present in the second part of our article.

It is known that connected sums of manifolds equipped with positive scalar curvature metrics admit such metrics as well [18, 42]. Also known is that positive scalar curvature metric is stable under codimension $q \geq 3$ surgeries [18, 42]. These results imply that the connected sums (1) and (2) admit positive scalar curvature metrics with respect to their standard smooth structures. Importantly, it means that stable cohomotopy Seiberg-Witten invariants of the connected sums of 4-manifolds given in (1) and (2) above, equipped with standard smooth structures, vanish. This fact, together with the new non-vanishing theorem [28] and Theorem B, allows us to prove the existence of pairwise homeomorphic but not diffeomorphic 4-manifolds *with trivial Seiberg-Witten invariants*. Namely, we get *exotic* copies of standard 4-manifolds which are connected sums of $\mathbb{CP}^2, \overline{\mathbb{CP}}^2, S^1 \times S^3, Y_p$, with trivial Seiberg-Witten invariants but non-trivial stable cohomotopy Seiberg-Witten invariants.

Corollary 2 *For* $i = 1, 2, 3$*, let* X_i *be any one of the* 4*-manifolds given in Theorem B. Then any connected sum* $\#_{i=1}^{j} X_i$ *admits an exotic smooth structure, for* $j = 2, 3$*.*

Moreover, by combining Theorem D in [28] with Theorems A and B of our paper, we also obtain

Corollary 3 *Let* X *be any closed, simply connected, non-spin, symplectic* 4*-manifold with* $b^+ \equiv 3 \pmod 4$*. For* $i = 1, 2$*, let* X_i *be any one of the* 4*-manifolds given in Theorem A or Theorem B. Then any connected sum* $X\#\left(\#_{i=1}^{j} X_i\right)$ *admits an exotic smooth structure, for* $j = 1, 2$*.*

Examples of closed non-spin and simply-connected 4-manifolds (which necessarily satisfy $b^+ \equiv 3 \pmod 4$) can be pulled out from the large collections of [1], or from earlier works of various authors in this direction. (See for instance Gompf's pioneer work [15].)

Another main application we will give regards the Ricci flow solutions on smooth 4-manifolds, and is discussed in Section 3. This is tightly related to Conjecture 1.8 of Fang, Zhang and Zhang in [12], as we will explain below. Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$. *The normalized Ricci flow* on X the following evolution equation [21]:

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}_g + \frac{2}{n} \bar{s}_g g, \tag{3}$$

where Ric_g is the Ricci curvature of the evolving Riemannian metric g, $\bar{s}_g := \int_X s_g d\mu_g / \mathrm{vol}_g$ and s_g denotes the scalar curvature of the evolving Riemannian metric g, $\mathrm{vol}_g := \int_X d\mu_g$ and $d\mu_g$ is the volume measure with respect to g. A solution $\{g(t)\}$, $t \in [0,T)$ of (3) on X is called *non-singular* [22] if $T = \infty$ and if the Riemannian curvature tensor $Rm_{g(t)}$ of $g(t)$ satisfies

$$\sup_{X\times[0,T)} |Rm_{g(t)}| < \infty.$$

In [22], Hamilton classified non-singular solutions to (3) on 3-manifolds. This work played an important role in understanding long-time behavior of solutions of the Ricci flow on 3-manifolds. In [12], Fang, Zhang and Zhang also studied the properties of non-singular solutions to (3) in higher dimensions. One of fundamental discoveries due to [12] is that, under a certain condition on the scalar curvature, the existence of the non-singular solutions of (3) brings constraints on the topology of the 4-manifold, and in particular on its Euler characteristic and signature:

$$2\mathrm{e}(X) - 3|\mathrm{sign}(X)| \geq 0.$$

This can be seen as a generalization of Hitchin-Thorpe inequality [24, 45] to Ricci flow case. Based on this fact, the authors proposed a conjecture. To state their conjecture precisely, we need to recall the definition of Perelman's $\bar{\lambda}$ invariant [40, 41]. Let g be any Riemannian metric on a closed oriented smooth manifold X with dimension $n \geq 3$. Consider the least eigenvalue λ_g of the elliptic operator $4\Delta_g + s_g$, where s_g denotes the scalar curvature of g, and $\Delta = d^*d = -\nabla\cdot\nabla$ is the positive-spectrum Laplace-Beltrami operator associated with g. λ_g can be expressed in terms of Raleigh quotients as

$$\lambda_g = \inf_u \frac{\int_X \left[s_g u^2 + 4|\nabla u|^2\right] d\mu}{\int_M u^2 d\mu},$$

where the infimum is taken over all smooth, real-valued functions u on X. Consider the the scale-invariant quantity $\lambda_g(vol_g)^{2/n}$, where $vol_g = \int_M d\mu_g$ denotes the total volume of (X, g). By taking the supremum of $\lambda_g(vol_g)^{2/n}$ over the space of all Riemannian metrics, we define Perelman's $\bar{\lambda}$ invariant of X:

$$\bar{\lambda}(X) = \sup_g \lambda_g(vol_g)^{2/n}. \tag{4}$$

The Fang-Zhang-Zhang conjecture can be stated as follows:

Conjecture 4 ([12]) *Let X be a closed oriented smooth Riemannian 4-manifold with $||X|| \neq 0$ and $\bar{\lambda}(X) < 0$, where $||X||$ denotes Gromov's simplicial volume.*

Suppose that there is a non-singular solution to the normalized Ricci flow on X*. Then the following holds:*

$$2e(X) - 3|sign(X)| \geq \frac{1}{1296\pi^2}||X||. \tag{5}$$

In this article, we refer to this conjecture as the *FZZ conjecture* in short. To the best of our knowledge, the FZZ conjecture remains open. In connection with this conjecture, the following problem arises naturally:

Problem 5 *Let* X *be a closed oriented smooth* 4*-manifold with* $||X|| \neq 0$, $\bar{\lambda}(X) < 0$ *and satisfying the inequality (5). Then, is there always a non-singular solution to the normalized Ricci flow on* X*?*

This is nothing but the converse of Conjecture 4. In the current article, we shall prove:

Theorem C *Let* X_m *be a BF-admissible closed oriented smooth* 4*-manifold and consider the following connected sum:*

$$M_{g,h,j}^{\ell_1,\ell_2} := (\#_{m=1}^{j} X_m)\#(\Sigma_h \times \Sigma_g)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{CP}^2},$$

where $j = 1, 2$*,* $\ell_1, \ell_2 \geq 1$*, and* $g, h \geq 3$ *are odd integers. Then, there are infinitely many sufficiently large integers* g, h, ℓ_1, ℓ_2 *for which* $M_{g,h,j}^{\ell_1,\ell_2}$ *has the following properties.*

1. X *has* $||X|| \neq 0$ *and satisfies the strict case of the inequality (5):*

$$2e(X) - 3|sign(X)| > \frac{1}{1296\pi^2}||X||.$$

2. X *admits at least one smooth structure for which Perelman's* $\bar{\lambda}$ *invariant is negative and there is no non-singular solution to the normalized Ricci flow for any initial metric.*

Since there are infinitely many BF-admissible closed 4-manifolds by Theorems A and B, as a corollary to Theorem C, we see that:

Corollary 6 *The converse of the FZZ conjecture fails to hold for vast families of* 4*-manifolds.*

In Subsection 3.4, we will propose a stronger version of the Conjecture 4; see Conjecture 26 stated there. We shall moreover derive results analagous to Theorem C; see Theorems D.

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2 Families of 4-manifolds satisfying BF-axioms

In this section, we will be proving Theorems A and B, which were stated in the Introduction.

2.1 Logarithmic transforms and Luttinger surgeries

Let L be an embedded self-intersection zero 2-torus in a 4-manifold X with oriented tubular neighborhood $N(L)$. A *framing* of $N(L)$ is a choice of an orientation-preserving diffeomorphism $\xi : N(L) \to D^2 \times T^2$, giving an identification

$$H_1(\partial(X \setminus N(L))) \cong H_1(L) \oplus \mathbb{Z}, \quad (6)$$

where the last summand is generated by a positively oriented meridian μ_L of L. We can construct a new 4-manifold $X' = X \setminus N(T) \cup_\phi D^2 \times T^2$ using a diffeomorphism $\phi : \partial(T^2 \times D^2) \to \partial N(L)$. This diffeomorphism is uniquely determined up to isotopy by the homology class

$$\phi_*[\partial D^2] = p[\mu_L] + q[S^1_\lambda] \, ,$$

where S^1_λ is a *push-off* of a primitive curve λ in L by the chosen framing ξ. To sum up, the result of the surgery is determined by the torus L, the framing ξ, the *surgery curve* λ and *the surgery coefficient* $p/q \in \mathbb{Q} \cup \{\infty\}$. This data is encoded in the notation $X(L, \lambda, p/q)$ whenever the framing is clear from the context. The operation producing $X' = X(L, \lambda, p/q)$ is called the *(generalized) logarithmic* p/q *transform of* X *along* L —with surgery curve λ and framing ξ, which we will denote by $(L, \lambda, p/q)$.

If (X, ω_X) is a symplectic manifold and L is a Lagrangian torus in X, then L admits a *Weinstein neighborhood* $N(L)$, which is a tubular neighborhood of L equipped with a canonical framing. This framing, called the *Lagrangian* *framing* here, is characterized by the unique property that $x \times T^2$, for any $x \in D^2$, corresponds to a Lagrangian submanifold of X under it. Let ξ be the Lagrangian framing and S^1_λ be the *Lagrangian push-off* of λ, i.e the push-off of λ in this framing. The $(L, \lambda, 1/q)$ surgery with these choices can be performed *symplectically*, providing us with —a deformation class of— a symplectic form $\omega_{X'}$ on $X' = X(L, \lambda, p/q)$ that agrees with ω_X on the complement of $N(L)$ [3]. This special logarithmic transform is referred as *Luttinger surgery.*

The classical topological invariants of 4-manifolds we are interested in this article change under logarithmic transforms (and in particular under Luttinger surgeries) as follows: Euler characteristic and signature of X' and X are the same, yet their spin types may differ depending on the choice of L and the surgery. It follows that when μ_L is nullhomologous and S^1_λ is homologically essential in $X \setminus N(L)$, we have $b_1(X') = b_1(X) - 1$ and $b_2(X') = b_2(X) - 2$. On the other hand, when both S^1_λ and L are nullhomologous in $X \setminus N(L)$,

$$H_1(X(L, \lambda, p/q); \mathbb{Z}) = H_1(X; \mathbb{Z}) \oplus \mathbb{Z}/p\mathbb{Z}\,.$$

Lastly, applying the Seifert-Van Kampen theorem, we get:

$$\pi_1(X(L, \lambda, p/q)) = \pi_1(X \setminus N(T))/\langle [\mu_L]^p [S^1_\lambda]^q = 1 \rangle. \tag{7}$$

It follows from the very definition that a logarithmic transform operation can be reversed, by performing a logarithmic transform along the *core torus* of the surgery that now lies in $X' = X(L, \lambda, p/q)$ by an appropriate choice of the surgery curve and the surgery coefficient. It is an easy exercise to see that, the same holds true in the symplectic setting; i.e. a Luttinger surgery can be reversed to obtain back the original symplectic 4-manifold. In this case, we will call it *undoing* the corresponding logarithmic transform or Luttinger surgery.

In what follows, we will mainly be interested in Luttinger surgeries so as to conclude that the resulting 4-manifolds we obtain satisfy the first assumption of Definition 1. Namely, we will be using the canonical class Γ_X associated to the resulting symplectic form, so that

$$SW_X(\Gamma_X) \equiv 1 \pmod 2\,, \text{and } c_1^2(\mathcal{L}_{\Gamma_X}) = 2\mathrm{e}(X) + 3\mathrm{sign}(X).$$

The rest of the assumptions will be seen to be satisfied merely by looking at the topological effect of the underlying logarithmic transforms.

2.2 Surgered product manifolds

Let X_0 be the product of two Riemann surfaces Σ_g and Σ_h, equipped with the product symplectic form. The second homology group $H_2(X_0)$ is generated by the homology classes of Σ_g, Σ_h and the Lagrangian tori $a_i \times c_j$, $a_i \times d_j$, $b_i \times c_j$, $b_i \times d_j$, $i = 1, \ldots, g$ and $j = 1, \ldots, h$, where a_i, b_i and c_j, d_j are the symplectic pairs of homology generators of the surfaces Σ_g and Σ_h, respectively. Assume that X_1 is obtained from X_0 via Luttinger surgeries along some of these homologically essential Lagrangian tori in X_0, such that: Each surgery is performed with surgery curve equal to one of a_i, b_i, c_j, d_j carried on the torus and with surgery coefficient equal to $1/n$ with respect to the Lagrangian framing, for some $n \in \mathbb{Z}$. In the present article, we shall call these new symplectic manifolds *surgered product manifolds.* Then we have

Lemma 7 *All surgered product manifolds obtained from $\Sigma_g \times \Sigma_h$ with $b_1 > 0$, are BF-admissible, for g, h are positive odd integers.*

Proof. Assume g and h are both odd positive integers. Since

$$b^+(\Sigma_g \times \Sigma_h) = 1 + 2gh, \text{ and } b_1(\Sigma_g \times \Sigma_h) = 2(g + h),$$

the difference $b^+ - b_1 \equiv 1 + 2(gh - g - h) \equiv 3 \pmod 4$. If we perform a torus surgery along any one of the product Lagrangian tori with the surgery curve equals any one of the homology generators (namely a_i, b_i, c_j or d_j for $i = 1, \ldots g$, $j = 1, \ldots, h$) and the surgery coefficient equals $1/n$ with respect to the Lagrangian framing, then b_1 drops by one, as seen from the equation (6). Note that we can just compute b_1 in $\mathbb{Q}$-coefficients so n can be any integer here.

Since the torus surgery does not change the Euler characteristic, b_2 of the new manifold we get drops by two. Moreover, what dies in the new homology is nothing but the homology class of the torus we performed the surgery along as well as the homology class of the torus dual to it. (A detailed analysis of this fact can be found in [25].) These Lagrangian tori made up a hyperbolic pair in the second homology of the original manifold, so we see that each one of b^+ and b^- drop by one. Hence the difference $b^+ - b_1$ remains the same and equals to $3 \pmod 4$ after the surgery, satisfying the second condition in Definition 1.

Now, let (X, ω_X) be the resulting symplectic 4-manifold obtained by a sequence of Luttinger surgeries of this sort in X_0. To guarantee that $b^+(X) > 1$, one just should not get taken by the heat of this process and kill all pairs of Lagrangian tori. It suffices to leave one such pair; $\Sigma_g \times \{pt\}$ and $\{pt\} \times \Sigma_h$ still descend to the new symplectic manifold as a hyperbolic pair, and together with another pair of Lagrangian tori we get $b^+(X_1) > 1$ as required.

For the class Γ_X take any almost complex structure compatible with the symplectic form, on which Seiberg-Witten invariant evaluates as 1 (and thus equals 1 (mod 2)) by Taubes' celebrated work in [44]. This particularly implies that the first condition in Definition 1 is satisfied.

The new set of generators for $H_1(X)$ is given by all a_i, b_i, c_j or d_j for $i = 1, \dots g$, $j = 1, \dots, h$ except for the used surgery curves. Now note that $a_p \times b_q$ and $c_r \times d_s$ for $p, q = 1, \dots, g$ and $r, s = 1, \dots, h$ (whichever still exist) are all trivial in $H_2(X)$. On the other hand all other possible products were prescribing Lagrangian tori in the symplectic manifold X_0. Since the canonical class of X_0 can be supported away from all these tori [3], these tori are still Lagrangian in X. Thus, if we choose Γ_X as an almost complex structure compatible with the symplectic form on X, then the evaluation $\mathfrak{S}^{ij}(\Gamma_X) := \frac{1}{2} < c_1(\Gamma_X) \cup \mathfrak{e}_i \cup \mathfrak{e}_j, [X] >$ is either trivially zero to begin with or is equal to evaluating ω on a Lagrangian torus, and thus, vanishes in all possible cases, satisfying the third condition in Definition 1. ∎

In addition to the manifolds of the type $\Sigma_g \times \Sigma_h$ for g, h odd, it was observed in [28] that a primary Kodaira surface is also BF-admissible. Both of these families of manifolds are indeed subfamilies of surgered manifolds, as we now show for the non-trivial case: [1]

Lemma 8 *A primary Kodaira surface* K *is a surgered product manifold. In particular,* K *is BF-admissible.*

Proof. Take $g = h = 1$ and perform one Luttinger surgery along any one of the homologically essential tori listed above. Without loss of generality we can assume that this torus is $a \times c$ (where we drop the subindices as $g = h = 1$). The resulting manifold can be described by the dimensionally reduced Kirby diagram given below. In the diagram one depicts $X_0 = T^2 \times T^2$ as $S^1 \times T^3$ where the first S^1 component corresponds to a, and not drawn. Then since the diagram and the surgery are set in an S^1 invariant way, the Luttinger surgery amounts to performing a Dehn surgery along c with coefficient n in the T^3 component [2]. The resulting diagram describes the smooth type of a primary Kodaira surface.

[1] Tian-Jun Li has informed us that this observation was known to him. Also see [25].

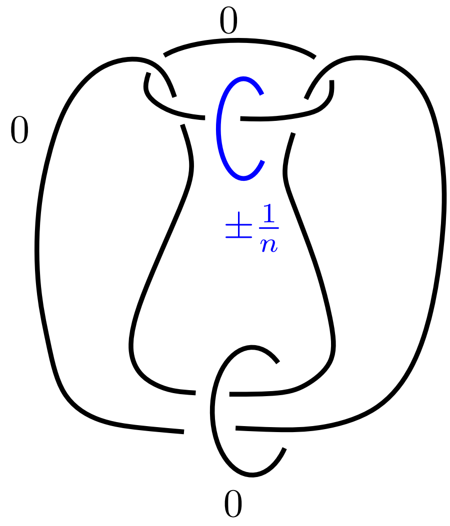

Figure 1: An S^1 invariant surgery diagram for a primary Kodaira surface.

Another way to see this is through the classification of Lagrangian torus bundles over tori (see [14]). The projection onto $\mathfrak{b} \times \mathfrak{d}$ describes a Lagrangian torus bundle on T^4 equipped with the product symplectic form (i.e. the sum of the pullbacks of the volume forms on tori $\mathfrak{a} \times \mathfrak{b}$ and $\mathfrak{c} \times \mathfrak{d}$). The reader can verify that the Luttinger surgery along $\mathfrak{a} \times \mathfrak{c}$ in question yields a new Lagrangian torus bundle over a torus, where the fiber now is necessarily inessential in homology. As the result is a symplectic 4-manifold admitting a Lagrangian torus bundle over a torus, it is a primary Kodaira surface. ∎

Theorem A now follows from Lemmas 7 and 8.

2.3 Families obtained from surgered product manifolds

We are now going to look at large families of 4-manifolds constructed using solely the surgered products as building blocks. Such families, spanning a large portion of the geography plane were obtained in [1]:

Theorem 9 (Theorem A in [1]) *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *denote integers satisfying* $2\mathfrak{a}+3\mathfrak{b} \geq 0$*, and* $\mathfrak{a}+\mathfrak{b} \equiv 0 \pmod 4$*. If, in addition,* $\mathfrak{b} \leq -2$*, then there exists a simply connected minimal symplectic* 4*-manifold with Euler characteristic* $\mathfrak{a}$ *and signature* $\mathfrak{b}$ *and odd intersection form, except possibly for* $(\mathfrak{a}, \mathfrak{b})$ *equal to* $(7, -3)$*,* $(11, -3)$*,* $(13, -5)$*, or* $(15, -7)$*.*

Note that the missing four lattice points given in the statement, using the minimal symplectic $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ constructed by Akhmedov and Park, can be realized by the same methods of [1], as shown in [4]. For the lack of a better name, we will call all these manifolds as *ABBKP manifolds* in short. A close look at these examples show that they are all obtained from surgered product manifolds via a couple of operations. Namely:

(1) Symplectic blow-ups at points on the symplectic surfaces $\Sigma_g \times \{pt\}$ or $\{pt\} \times \Sigma_h$ in the surgered product manifolds; and

(2) Symplectic fiber sums along *symplectic* surfaces which are obtained from copies of $\Sigma_g \times \{pt\}$, $\{pt\} \times \Sigma_h$ and exceptional spheres that might have been introduced during blow-ups.

Said differently, these manifolds are obtained by using symplectic building blocks $\Sigma_g \times \Sigma_h$ —where g and h are *not* necessarily odd, the above two operations, *and* Luttinger surgeries with coefficients ± 1 are performed along the product Lagrangian tori contained in them. This is because these Lagrangian tori are away from the standard symplectic surfaces $\Sigma_g \times \{pt\}$ or $\{pt\} \times \Sigma_h$, and remain Lagrangian after blow-ups of fiber sums. Therefore, one can perform the above two operations and the Luttinger surgeries in any order to get the resulting symplectic 4-manifold.

To meet the first and second conditions in Definition 1, we only deal with those X with $b^+(X) \equiv 3 \pmod 4$. Since $b_1(X) = 0$, the third condition in Definition 1 is satisfied vacuously for these manifolds. Now, if we *undo* any of the Luttinger surgeries, from our previous arguments in the proof of Lemma 7 we see that we re-introduce the hyperbolic pair of Lagrangian tori in the new resulting symplectic manifold, but all the conditions in Definition 1 are still satisfied. Hence we see that:

Theorem 10 *If one undoes any collection of the Luttinger surgeries involved in the construction of any one of the ABBKP manifold with $b^+ \equiv 3 \pmod 4$, the resulting manifold meets all the conditions in Definition 1.*

Undoing these surgeries in simply-connected end products will re-introduce b_1 in a straightforward fashion. The change in fundamental group however is more subtle, and is to our interest mostly when we only undo one of the surgeries to get manifolds with fundamental group $\mathbb{Z}$ and perform the last surgery with general Luttinger surgery coefficient $\pm 1/m$ instead of ± 1 to get $\mathbb{Z}_m$, for which we can use homeomorphism criteria given by the following theorems:

Theorem 11 (Hambleton-Teichner [20], see also [33].) *Let* X *be a smooth closed oriented* 4*-manifold with infinite cyclic fundamental group.* X *is classified up to homeomorphism by the fundamental group, the intersection on* $H_2(X, \mathbb{Z})/\mathrm{Tors}$ *and the* w_2*-type. If in addition,* $b_2(X) - |sign(X)| \geq 6$*, then* X *is homeomorphic to the connected sum of* $S^1 \times S^3$ *with a unique closed simply connected* 4*-manifold. In particular,* X *is determined up to homeomorphism by its second Betti number* $b_2(X)$*, its signature* $\tau(X)$ *and its* w_2*-type. Particularly,* X *is either spin or non-spin depending on the parity of its intersection form.*

Theorem 12 (Hambleton-Kreck [19]) *Let* X *be a closed smooth oriented* 4*-manifold with finite cyclic fundamental group. Then* X *is classified up to homeomorphism by the fundamental group, the intersection form on* $H_2(X; \mathbb{Z})/\mathrm{Tors}$*, and the* ω_2*-type. Moreover, any isometry of the intersection form can be realized by a homeomorphism.*

A 0-surgery along $\{pt\} \times S^1$ in $L(p, 1) \times S^1$ yields a manifold with fundamental group $\mathbb{Z}_p$, which has the smallest homology among all other 4-manifolds of the same fundamental group, which we denote by Y_p. A bi-product of the above discussion gives rise to Theorem B:

Proof. [Theorem B] In [1], a key ingredient in the constructions were the *telescoping triples.* We recall the definition of a telescoping triple here for the convenience of reader: An ordered triple (X, T_1, T_2) where X is a symplectic 4-manifold and T_1, T_2 are disjointly embedded Lagrangian tori is called a telescoping triple if

(i) The tori T_1, T_2 span a 2-dimensional subspace of $H_2(X; \mathbb{R})$.

(ii) $\pi_1(X) = \mathbb{Z} \oplus \mathbb{Z}$ and the inclusion induces an isomorphism $\pi_1(X \setminus (T_1 \cup T_2)) \to \pi_1(X)$, which in particular implies that the meridians of the T_i are trivial in $\pi_1(X \setminus (T_1 \cup T_2))$.

(iii) The image of the homomorphism induced by inclusion $\pi_1(T) \to \pi_1(X)$ is a summand $\mathbb{Z}$ in $\pi_1(X)$.

(iv) The homomorphism induced by inclusion $\pi_1(T_2) \to \pi_1(X)$ is an isomorphism.

Each ABBKP manifold X' is obtained using various telescoping triples. In particular, X' can be viewed as obtained from a telescoping triple (X, T_1, T_2) (say the 'last' telescoping triple involved in the construction) after a ± 1

Luttinger surger along T_2. The very properties of a telescoping triple implies that undoing the Luttinger surgery along the core-torus that descends from T_2 hands us back a symplectic 4-manifold Z with fundamental group $\mathbb{Z}$. Now if one performs a Luttinger surgery along T_2 in Z with the same surgery curve but with surgery coefficient $1/p$ instead, from Seifert-Van Kampen calculation we get a symplectic 4-manifold Z_p with fundamental group $\mathbb{Z}_p$. (Note that the sign of the surgery does not effect the resulting fundamental group, so it is not relevant to our discussion here.)

We claim that the manifolds Z and Z_p constructed for each ABBKP manifold X make up the families

$$\alpha\mathbb{C}P^2\#\beta\overline{\mathbb{C}P^2}\#(S^1 \times S^3) \text{ and}$$

$$(\alpha-1)\mathbb{C}P^2\#(\beta-1)\overline{\mathbb{C}P^2}\#Y_p\,,$$

respectively. From Theorem 9, there is an X with $a = e(X)$, $b = \text{sign}(X)$ satisfying $a+b \equiv 0 \pmod 8$, and $b \leq -2$. (These constitute the 'half' of the ABBKP manifolds, since we require $a + b \equiv 0 \pmod 8$ instead of $a + b \equiv 0 \pmod 4$.) Therefore, the Euler characteristic and signature of Z and Z_p are also equal to a and b, respectively. Clearly, both are non-spin smooth 4-manifolds, and satisfy $b_2(X) - |\text{sign}(X)| \geq 6$. Now, $\pi_1(Z) = H_1(Z) = \mathbb{Z}$ and $\pi_1(Z_p) = H_1(Z_p) = \mathbb{Z}_p$ for p odd, lands Z in the same homeomorphism class of $(a+b)/2\mathbb{C}P^2\#(a-b)/2\overline{\mathbb{C}P^2}\#(S^1 \times S^3)$ by Theorem 11 and Z_p in $(a+b/2-1)\mathbb{C}P^2\#(a-b/2-1)\overline{\mathbb{C}P^2}\#Y_p$ by Theorem 12, respectively.

To prove that the manifolds Z and Z_p are irreducible, we recall that they can equivalently be obtained from surgered products via two operations (1) and (2) discussed above. There are two key observations made in [1] and [4] to conclude the minimality of ABBKP manifolds: First of all, the surgered products used in these constructions are minimal. After blow-ups minimality is lost in the pieces, however, the fiber sums that follow are performed along symplectic surfaces that intersect the new exceptional spheres in the way that Usher's theorem on minimality of symplectic fiber sums [47] can be employed to conclude that the resulting symplectic 4-manifold is minimal. The same observations hold true when one of the Luttinger surgeries goes undone, since the only difference now surfaces in one of the surgered products containing the corresponding Lagrangian torus being obtained from a product of Riemann surfaces with one less Luttinger surgery (and thus yielding a non-rational surface bundle over a non-rational surface, which has no π_2). Hence, both Z and Z_p are minimal symplectic 4-manifolds with residually finite fundamental groups. By [23], they are irreducible.

Lastly, our claim that manifolds Z and Z_p satisfy the BF-axioms, follow from Theorem 10. ∎

Remark In [1] many more possible lattice points in the geography plane for sign ≤ 4 were realized by minimal symplectic 4-manifolds, leaving out about 280 lattice points. Moreover, the small manifolds constructed by Akhmedov and Park in [4] leads to a slight enlargement of this region spanned by the minimal symplectic 4-manifolds. These manifolds can also be used to enlarge our families obtained in Theorem B—a similar discussion can be found in [46]. Nevertheless, we are content with the vast families we have got for the applications that will follow in the next chapter, and therefore will not discuss these slight extensions here. □

3 The Ricci flow and the FZZ conjecture

3.1 Asymptotic behavior of the Ricci curvature

Inspired by works of Cao [11] and Li [36], one parameter family $\bar{\lambda}_k$ of smooth invariants, where $k \in \mathbb{R}$, was introduced in [28]. It is called $\bar{\lambda}_k$ invariant. Let X be a closed oriented Riemannian manifold with dimension ≥ 3. Then, recall the following variant [36, 39] $\mathcal{F}_k : \mathcal{R}_X \times C^\infty(X) \to \mathbb{R}$ of the Perelman's $\mathcal{F}$-functional [40]:

$$\mathcal{F}_k(g, f) := \int_X \Big(k s_g + |\nabla f|^2\Big) e^{-f} d\mu_g, \tag{8}$$

where k is a real number $k \in \mathbb{R}$. We shall call this $\mathcal{F}_k$-functional. Notice that $\mathcal{F}_1$-functional is nothing but Perelman's $\mathcal{F}$-functional. Li [36] pointed out that all functionals $\mathcal{F}_k$ with $k \geq 1$ have the monotonicity properties under a certain coupled system of Ricci flow. As was already mentioned in [36, 29] essentially, for a given metric g and $k \in \mathbb{R}$, there exists a unique minimizer of the $\mathcal{F}_k$-functional under the constraint $\int_X e^{-f} d\mu_g = 1$. In fact, by using a direct method of the elliptic regularity theory, one can see that the following infimum is always attained:

$$\lambda(g)_k := \inf_f \{\mathcal{F}_k(g, f) \mid \int_X e^{-f} d\mu_g = 1\}.$$

Notice that $\lambda(g)_k$ is nothing but the least eigenvalue of the elliptic operator $4\Delta_g + ks_g$. It is then natural to introduce the following invariant [28] which is called $\bar{\lambda}_k$ invariant of X:

$$\bar{\lambda}_k(X) = \sup_{g \in \mathcal{R}_X} \lambda(g)_k (vol_g)^{2/n}.$$

It is clear that $\bar{\lambda}_1(X) = \bar{\lambda}(X)$ holds. Then we have:

Theorem 13 ([28]) *For $m = 1, 2, 3$, let X_m be a BF-admissible 4-manifold and set as $c_1^2(X_m) = 2e(X_m) + 3sign(X_m)$. Suppose that N is a closed oriented smooth 4-manifold with $b^+(N) = 0$. Consider a connected sum $M := \left(\#_{m=1}^n X_m\right) \# N$, where $n = 2, 3$. Suppose moreover that $\sum_{m=1}^n c_1^2(X_m) > 0$ holds, where $n = 2, 3$. Then, for $n = 2, 3$ and any real number $k \geq \frac{2}{3}$, $\bar{\lambda}_k$ invariant of a connected sum $M := (\#_{m=1}^n X_m) \# N$ satisfies*

$$\bar{\lambda}_k(M) \leq -4k\pi \sqrt{2 \sum_{m=1}^n c_1^2(X_m)} < 0.$$

As a corollary of Theorem 13, we obtain:

Corollary 14 *Let X_m be a BF-admissible closed oriented smooth 4-manifold and consider the a connected sum:*

$$M := (\#_{m=1}^j X_m) \# (\Sigma_h \times \Sigma_g) \# \ell_1 (S^1 \times S^3) \# \ell_2 \overline{\mathbb{CP}^2},$$

where $j = 1, 2$, $\ell_1, \ell_2 \geq 1$. And $g, h \geq 3$ are odd integers such that $c_{g,h}^j := \sum_{m=1}^j (2e(X_m) + 3sign(X_m)) + 4(1-h)(1-g) > 0$. Then, for any real number $k \geq \frac{2}{3}$, $\bar{\lambda}_k$ invariant of the connected sum M is given by

$$\bar{\lambda}_k(M) \leq -4k\pi \sqrt{2c_{g,h}^j} < 0. \tag{9}$$

Proof. By Theorem A, the product $\Sigma_h \times \Sigma_g$ of Riemann surface with odd genus is BF-admissible. Hence, Theorem 13 implies the bound (9). ∎

We also have:

Lemma 15 *Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$ and assume that there is a positive real number k such that the $\overline{\lambda}_k(X) < 0$. If there is a solution $\{g(t)\}$, $t \in [0, T)$, to the normalized Ricci flow, then*

$$\hat{s}_{g(t)} := \min_{x \in X} s_{g(t)}(x) \leq \frac{\overline{\lambda}_k(X)}{k(vol_{g(0)})^{2/n}} < 0,$$

where we define as $\hat{s}_g := \min_{x \in X} s_g(x)$ for a given Riemannian metric g.

Proof. The case where $k = 1$ is proved in [26]. Though the proof is similar to the case, we shall include the proof for the reader. Let $\{g(t)\}$ be any solution to the normalized Ricci flow on X. Notice that $\lambda_{g(t)}$ can be expressed in terms of Raleigh quotients as

$$\lambda_{g(t)} = \inf_{u} \frac{\int_X \left[ks_{g(t)}u^2 + 4|\nabla u|^2\right] d\mu_{g(t)}}{\int_X u^2 d\mu_{g(t)}},$$

where the infimum is taken over all smooth real-valued functions u on X. Therefore we have

$$\begin{aligned} \lambda_{g(t)} &= \inf_{u} \frac{\int_X \left[ks_{g(t)}u^2 + 4|\nabla u|^2\right] d\mu_{g(t)}}{\int_X u^2 d\mu_{g(t)}} \\ &\geq \inf_{u} \frac{\int_X \left[k\widehat{s}_{g(t)}u^2 + 4|\nabla u|^2\right] d\mu_{g(t)}}{\int_X u^2 d\mu_{g(t)}} \\ &\geq k\widehat{s}_{g(t)}\Big(\inf_{u} \frac{\int_X u^2 d\mu_{g(t)}}{\int_X u^2 d\mu_{g(t)}}\Big) = k\widehat{s}_{g(t)}. \end{aligned}$$

Hence $\lambda_{g(t)} \geq k\widehat{s}_{g(t)}$ holds. By the very definition of $\overline{\lambda}_k$ invariant, we have $\overline{\lambda}_k(X) \geq \lambda_{g(t)}(vol_{g(t)})^{2/n}$. We therefore get $\overline{\lambda}_k(X) \geq k\widehat{s}_{g(t)}(vol_{g(t)})^{2/n}$. Since the normalized Ricci flow preserves the volume of the solution, we have $vol_{g(t)} = vol_{g(0)}$. Hence, we get $\overline{\lambda}_k(X) \geq k\widehat{s}_{g(t)}(vol_{g(0)})^{2/n}$. ∎

Theorem 13 and 15 tell us that the following result holds:

Theorem 16 *Under the same assumption with Theorem 13, a solution to the normalized Ricci flow satisfies the following bound:*

$$\widehat{s}_{g(t)} := \min_{x \in M} s_{g(t)}(x) \leq -\Big(\frac{4\pi}{(vol_{g(0)})^{1/2}}\sqrt{2\sum_{m=1}^{n} c_1^2(X_m)}\Big) < 0. \tag{10}$$

Notice that the right hand side of the bound (10) is a negative constant of independent of both $x \in M$ and t.

We also need to recall the following key result proved in [12]:

Lemma 17 ([12]) *Let X be a closed oriented Riemannian 4- manifold and assume that there is a long time solution $\{g(t)\}$, $t \in [0, \infty)$, to the normalized Ricci flow. Assume moreover that the solution satisfies $|s_{g(t)}| \leq C$ and $\widehat{s}_{g(t)} \leq$*

$-c < 0$ *where the constants* C *and* c *is independent of both* $x \in X$ *and time* $t \in [0, \infty)$. *Then, the trace-free part* $\overset{\circ}{r}_{g(t)}$ *of the Ricci curvature satisfies*

$$\int_0^\infty \int_X | \overset{\circ}{r}_{g(t)} |^2 d\mu_{g(t)} dt < \infty.$$

This result and Theorem 16 imply immediately the following result used to prove Theorem 20 stated below:

Theorem 18 *Under the same assumption with Theorem 13, any non-singular solution to the normalized Ricci flow on* M *satisfies*

$$\int_\ell^{\ell+1} \int_M | \overset{\circ}{r}_{g(t)} |^2 d\mu_{g(t)} dt \longrightarrow 0 \tag{11}$$

holds when $\ell \to +\infty$.

3.2 Obstruction

Let X be a closed oriented smooth 4-manifold with $b^+(X) \geq 2$. An element $\mathfrak{a} \in H^2(X, \mathbb{Z})/\text{torsion} \subset H^2(X, \mathbb{R})$ is called monopole class [32, 35, 27] of X if there exists a spin^c structure Γ_X with $c_1^{\mathbb{R}}(\mathcal{L}_{\Gamma_X}) = \mathfrak{a}$ which has the property that the corresponding Seiberg-Witten monopole equations have a solution for every Riemannian metric on X. Here $c_1^{\mathbb{R}}(\mathcal{L}_{\Gamma_X})$ is the image of the first Chern class $c_1(\mathcal{L}_{\Gamma_X})$ of the complex line bundle $\mathcal{L}_{\Gamma_X}$ in $H^2(X, \mathbb{R})$. It is known [27] that the non-triviality of BF_* implies the existence of monopole classes. LeBrun [34, 35] proved the existence of monopole classes implies several interesting curvature bounds, which have many beautiful differential geometric applications. By combining the new non-vanishing theorem proved in [28] with the curvature bounds of LeBrun, we obtain the following result:

Theorem 19 ([28]) *For* $m = 1, 2, 3$, *let* X_m *be a BF-admissible* 4*-manifold and set as* $c_1^2(X_m) = 2e(X_m) + 3sign(X_m)$. *Suppose that* N *is a closed oriented smooth* 4*-manifold with* $b^+(N) = 0$. *Consider a connected sum* $M := \left(\#_{m=1}^n X_m\right) \# N$, *where* $n = 2, 3$. *Then any Riemannian metric* g *on* M *satisfies*

$$\frac{1}{4\pi^2} \int_M \left(2|W_g^+|^2 + \frac{s_g^2}{24}\right) d\mu_g \geq \frac{2}{3} \sum_{m=1}^n c_1^2(X_m). \tag{12}$$

where W_g^+ *denote the self-dual Weyl curvature of* g.

On the other hand, the Chern-Gauss-Bonnet formula and the Hirzebruch signature formula for a closed oriented Riemannian 4-manifold X tell us that the following formulas hold for any Riemannian metric g on X:

$$\mathrm{sign}(X) = \frac{1}{12\pi^2}\int_X \Big(|W_g^+|^2 - |W_g^-|^2\Big) d\mu_g,$$

$$\mathrm{e}(X) = \frac{1}{8\pi^2}\int_X \Big(\frac{s_g^2}{24} + |W_g^+|^2 + |W_g^-|^2 - \frac{|\stackrel{\circ}{r}_g|^2}{2}\Big) d\mu_g,$$

where W_g^+ and W_g^- denote respectively the self-dual and anti-self-dual Weyl curvature of the metric g and $\stackrel{\circ}{r}_g$ is the trace-free part of the Ricci curvature of the metric g. By these formulas, we get the following:

$$2\mathrm{e}(X) + 3\mathrm{sign}(X) = \frac{1}{4\pi^2}\int_X \Big(2|W_g^+|^2 + \frac{s_g^2}{24} - \frac{|\stackrel{\circ}{r}_g|^2}{2}\Big) d\mu_g, \qquad (13)$$

Then we obtain:

Theorem 20 *Let N be a closed oriented smooth 4-manifold with $b^+(N) = 0$. For $m = 1, 2, 3$, For $m = 1, 2, 3$, let X_m be a BF-admissible 4-manifold.Assume also that $\sum_{m=1}^{n} c_1^2(X_m) > 0$ is satisfied, where $n = 2, 3$ and $c_1^2(X_m) = 2e(X_m) + 3sign(X_m)$. Then, on a connected sum $M := (\#_{m=1}^{n} X_m)\# N$, where $n = 2, 3$, there is no non-singular solution to the normalized Ricci flow for any initial metric if the following holds:*

$$4n - \Big(2e(N) + 3sign(N)\Big) > \frac{1}{3}\sum_{m=1}^{n} c_1^2(X_m). \qquad (14)$$

Proof. By (12), we obtain the following bound which holds for any Riemannian metric g on M:

$$\frac{1}{4\pi^2}\int_M \Big(2|W_g^+|^2 + \frac{s_g^2}{24}\Big) d\mu_g \geq \frac{2}{3}\sum_{m=1}^{n} c_1^2(X_m). \qquad (15)$$

Suppose now that there is a non-singular solution $\{g(t)\}$ to the normalized Ricci flow on M. Then, we have the following bound by (15)

$$\frac{1}{4\pi^2}\int_M \Big(2|W_{g(t)}^+|^2 + \frac{s_{g(t)}^2}{24}\Big) d\mu_{g(t)} \geq \frac{2}{3}\sum_{m=1}^{n} c_1^2(X_m). \qquad (16)$$

By (11) and (13), we are able to obtain

$$
\begin{aligned}
2e(M) + 3\mathrm{sign}(M) &= \lim_{\ell \longrightarrow \infty} \int_{\ell}^{\ell+1} \Big(2e(M) + 3\mathrm{sign}(M)\Big) dt \\
&= \lim_{\ell \longrightarrow \infty} \frac{1}{4\pi^2} \int_{\ell}^{\ell+1} \int_M \Big(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24} - \frac{|\overset{\circ}{r}_{g(t)}|^2}{2}\Big) d\mu_{g(t)} dt \\
&= \lim_{\ell \longrightarrow \infty} \frac{1}{4\pi^2} \int_{\ell}^{\ell+1} \int_M \Big(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24}\Big) d\mu_{g(t)} dt.
\end{aligned}
$$

This and the bound (16) imply

$$
\begin{aligned}
2e(M) + 3\mathrm{sign}(M) &= \lim_{\ell \longrightarrow \infty} \frac{1}{4\pi^2} \int_{\ell}^{\ell+1} \int_M \Big(2|W^+_{g(t)}|^2 + \frac{s^2_{g(t)}}{24}\Big) d\mu_{g(t)} dt \\
&\geq \lim_{\ell \longrightarrow \infty} \frac{2}{3} \int_{\ell}^{\ell+1} \sum_{m=1}^{n} c_1^2(X_m) dt = \frac{2}{3} \sum_{m=1}^{n} c_1^2(X_m).
\end{aligned}
$$

Since a direct computation tells us that $2e(M)+3\mathrm{sign}(M) = \sum_{m=1}^{n} c_1^2(X_m) - 4n + \Big(2e(N)+3\mathrm{sign}(N)\Big)$, the desired result now follows from the above bound by contraposition. ■

As a corollary of Theorem 20, we get:

Corollary 21 *For $m = 1, 2$, let X_m be a BF-admissible 4-manifold. Consider a connected sum*

$$M := (\#_{m=1}^{j} X_m) \# (\Sigma_g \times \Sigma_h) \# \ell_1 (S^1 \times S^3) \# \ell_2 \overline{\mathbb{C}P^2},$$

where $j = 1, 2$, $\ell_1, \ell_2 \geq 0$, and g, h are odd integers ≥ 1. Then there is no non-singular solution to the normalized Ricci flow on M if

$$4(j + \ell_1) + \ell_2 > \frac{1}{3}\Big(\sum_{m=1}^{j} 2e(X_m) + 3sign(X_m) + 4(1-h)(1-g)\Big).$$

Proof. Theorem A particularly tells us that $\Sigma_g \times \Sigma_h$ is BF-admissible. Notice also that we have $2e(N) + 3\mathrm{sign}(N) = 4 - 4\ell_1 - \ell_2$ by setting as $N := \ell_1(S^1 \times S^3) \# \ell_2 \overline{\mathbb{C}P^2}$. By taking $n = 3$ in the inequality (14), we have the desired result. ■

3.3 Proof of Theorem C

For the definition and the fundamental properties of Gromov's simplicial volume, see [16, 8]. In particular, notice that any simply connected manifold M satisfies $||M|| = 0$. We begin with:

Lemma 22 *Let* X_m *be a closed* 4*-manifold and consider a connected sum:*

$$M := (\#_{m=1}^{j} X_m)\#k(\Sigma_h \times \Sigma_g)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{C}P^2},$$

where $g, h \geq 2$*,* $j, k \geq 1$ *and* $\ell_1, \ell_2 \geq 0$*. Then the simplicial volume of* M *is given by*

$$||M|| = 24k(g-1)(h-1) + \sum_{m=1}^{j} ||X_m||. \tag{17}$$

On the other hand, we have

$$\begin{aligned} 2e(M) + 3sign(M) &= \Big(\sum_{m=1}^{j} 2e(X_m) + 3sign(X_m)\Big) + 4k(g-1)(h-1) \\ &- 4(j+k-1+\ell_1) - \ell_2, \\ 2e(M) - 3sign(M) &= \Big(\sum_{m=1}^{j} 2e(X_m) - 3sign(X_m)\Big) + 4k(g-1)(h-1) \\ &- 4(j+k-1+\ell_1) + 5\ell_2. \end{aligned}$$

Proof. It is known [16, 8] that the simplicial volume of the connected sum satisfies $||M_1\#M_2|| = ||M_1|| + ||M_2||$ Since it is also known that $||S^1 \times S^3|| = 0$ and $||\overline{\mathbb{C}P^2}|| = 0$ hold, we have $||M|| = k||\Sigma_h \times \Sigma_g|| + \sum_{m=1}^{j} ||X_m||$. On the other hand, by [10], $||\Sigma_h \times \Sigma_g|| = 24(g-1)(h-1)$ holds. Therefore, we have the formula (17). One can also deduce the formulas on $2e(M) + 3sign(M)$ and $2e(M) - 3sign(M)$ by direct computations. ∎

Lemma 23 *Let* X_m *be a closed oriented smooth* 4*-manifold and consider the following connected sum:*

$$M := (\#_{m=1}^{j} X_m)\#(\Sigma_h \times \Sigma_g)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{C}P^2},$$

where $j = 1, 2$*. For any pair* (g, h) *of positive integers* ≥ 2*, define the following positive number:*

$$\kappa(g,h) := 4(1-h)(1-g) - \frac{24(1-h)(1-g)}{1296\pi^2} > 0. \tag{18}$$

Then, there are infinitely many sufficiently large integers g, h, ℓ_1, ℓ_2 *for which the following three conditions are satisfied simultaneously:*

$$\sum_{m=1}^{j}\Big(2e(X_m) - 3sign(X_m)\Big) > -\kappa(g,h) + \frac{||X||}{1296\pi^2} + 4(j+\ell_1) - 5\ell_2, \quad (19)$$

$$\sum_{m=1}^{j}\Big(2e(X_m) + 3sign(X_m)\Big) > -\kappa(g,h) + \frac{||X||}{1296\pi^2} + 4(j+\ell_1) + \ell_2, \quad (20)$$

$$4(j+\ell_1)+\ell_2 > \frac{1}{3}\Big(\sum_{m=1}^{j}\Big(2e(X_m) + 3sign(X_m)\Big) + 4(1-h)(1-g)\Big), \quad (21)$$

where set as $||X|| := \sum_{m}^{j} ||X_m|| \in [0,\infty)$.

Proof. First of all, notice that the inequality (19) is always satisfied by taking sufficiently large ℓ_2 for any fixed ℓ_1, g, h. On the other hand, the inequality (20) is equivalent to

$$c_j + \kappa(g,h) - \frac{||X||}{1296\pi^2} > 4(j+\ell_1) + \ell_2, \quad (22)$$

where $c_j := \sum_{m=1}^{j}\Big(2e(X_m) + 3\text{sign}(X_m)\Big)$. Therefore, by (21) and (22), it is enough to prove that there exist infinitely many sufficiently large positive integers ℓ_1, ℓ_2, g, h satisfying

$$c_j + \kappa(g,h) - \frac{||X||}{1296\pi^2} > 4(j+\ell_1) + \ell_2 > \frac{1}{3}\Big(c_j + 4(1-h)(1-g)\Big). \quad (23)$$

We set as

$$A := c_j + \kappa(g,h) - \frac{||X||}{1296\pi^2}, \; B := \frac{1}{3}\Big(c_j + 4(1-h)(1-g)\Big),$$

namely, (23) is nothing but $A > 4(2+\ell_1)+\ell_2 > B$. Notice that both A and B can become sufficiently large positive integers by taking sufficiently large g or h. We also have

$$A - B = \frac{2}{3}c_j + \Big(\frac{8}{3} - \frac{24}{1296\pi^2}\Big)(1-h)(1-g) - \frac{||X||}{1296\pi^2}.$$

From this, we see that $A - B$ can become a large positive integer by taking large g or h. Since there are infinitely many choices of such g and h, we are able to conclude that there are also infinitely many ℓ_1, ℓ_2 satisfying $A > 4(j + \ell_1) + \ell_2 > B$. By taking sufficiently large g or h, we are also able to find a sufficiently large ℓ_2 satisfying the inequality (19), where notice that $\kappa(g, h) > 0$ and also that we can take as $4(j + \ell_1) - 5\ell_2 < 0$. ∎

Lemma 22 and Lemma 23 imply:

Proposition 24 *Let* X_m *be a closed oriented smooth* 4*-manifold and consider the a connected sum:*

$$M := (\#_{m=1}^{j} X_m) \# (\Sigma_h \times \Sigma_g) \# \ell_1 (S^1 \times S^3) \# \ell_2 \overline{\mathbb{C}P^2},$$

where $j = 1, 2$*. Then, there are infinitely many sufficiently large integers* g, h, ℓ_1, ℓ_2 *for which the following two conditions are satisfies simultaneously:*

$$2e(M) - 3|sign(X)| > \frac{||M||}{1296\pi^2}, \tag{24}$$

$$4(j + \ell_1) + \ell_2 > \frac{1}{3}\Big(\sum_{m=1}^{j} \Big(2e(X_m) + 3sign(X_m)\Big) + 4(1-h)(1-g)\Big). \tag{25}$$

Proof. Notice that (25) is nothing but (21). On the other hand, by Lemma 22, we have $2\mathrm{e}(M) + 3\mathrm{sign}(M) = (\sum_{m=1}^{j} 2\mathrm{e}(X_m) + 3\mathrm{sign}(X_m)) + 4(g-1)(h-1) - 4(j+\ell_1) - \ell_2$. By (17), we also obtain

$$\frac{||M||}{1296\pi^2} = \frac{24}{1296\pi^2}(g-1)(h-1) + \frac{1}{1296\pi^2}\sum_{m=1}^{j} ||X_m||.$$

Therefore, the inequality (20) is nothing but

$$2\mathrm{e}(M) + 3\mathrm{sign}(X) > \frac{||M||}{1296\pi^2}. \tag{26}$$

Similarly, since Lemma 22 tells us that $2\mathrm{e}(M) - 3\mathrm{sign}(M) = (\sum_{m=1}^{j} 2\mathrm{e}(X_m) - 3\mathrm{sign}(X_m)) + 4(g-1)(h-1) - 4(j+\ell_1) + 5\ell_2$, the inequality (19) is nothing but

$$2\mathrm{e}(M) - 3\mathrm{sign}(X) > \frac{||M||}{1296\pi^2}. \tag{27}$$

By (26) and (27), we obtain (24) as desired. ∎

We are now in a position to prove Theorem C: First of all, by (17) in the case where $k = 1$, we have $||M|| \neq 0$ for any $g, h > 1$. On the other hand, by Corollary 14, under $c^j_{g,h} > 0$, we have $\bar{\lambda}(M) < 0$. Notice that $c^j_{g,h} > 0$ is always satisfied for sufficiently large odd integers $g, h > 1$ because $c^j_{g,h} := \sum_{m=1}^{j}(2e(X_m)+3\text{sign}(X_m))+4(1-h)(1-g)$ holds. Moreover, Proposition 24 tells us that there are infinitely many sufficiently large integers g, h, ℓ_1, ℓ_2 for which (24) and (25) are satisfied simultaneously. By (24), M satisfies the strict case of the inequality (5). On the other hand, under (25), there is no non-singular solution to the normalized Ricci flow on M for any initial metric by Corollary 21. Hence, Theorem C follows.

On the other hand, we are able to prove a slight stronger version of Theorem C by taking a sequence of homotopy $K3$ surfaces. Let Y_0 be a Kummer surface with an elliptic fibration $Y_0 \to \mathbb{C}P^1$. Let Y_ℓ be obtained from Y_0 by performing a logarithmic transformation of order $2m + 1$ on a non-singular fiber of Y_0. Then, Y_m are simply connected spin manifolds with $b^+(Y_m) = 3$ and $b^-(Y_m) = 19$. By the Freedman classification [13], Y_m must be homeomorphic to a $K3$ surface. And Y_m is a Kähler surface with $b^+(Y_m) > 1$ and hence a result of Witten [48] tells us that $\pm c_1(Y_m)$ are monopole classes of Y_m for each m. Notice also that Y_m is BF-admissible. By using Y_m and Theorem C, we obtain:

Theorem 25 *Let X be a BF-admissible closed oriented smooth 4-manifold and consider the following connected sum:*

$$M^{\ell_1,\ell_2}_{g,h} := X\#K3\#(\Sigma_h \times \Sigma_g)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{C}P^2}, \tag{28}$$

where $\ell_1, \ell_2 \geq 1$, and $g, h \geq 3$ are odd integers. Then, there are infinitely many sufficiently large integers g, h, ℓ_1, ℓ_2 for which $M^{\ell_1,\ell_2}_{g,h}$ has the following properties.

1. *X has $||X|| \neq 0$ and satisfies the strict case of the inequality (5):*

$$2e(X) - 3|sign(X)| > \frac{1}{1296\pi^2}||X||.$$

2. *X admits infinitely many smooth structure for which the values of Perelman's $\bar{\lambda}$ invariants are negative and there is no non-singular solution to the normalized Ricci flow for any initial metric.*

Proof. First of all, notice that X has at least one monopole class $c_1(X)$ because X is BF-admissible. Consider the following connected sum which is homeomorphic to (28) for any m:

$$Z^{\ell_1,\ell_2}_{g,h}(m) := X \# Y_m \# (\Sigma_h \times \Sigma_g) \# \ell_1 (S^1 \times S^3) \# \ell_2 \overline{\mathbb{C}P^2}.$$

For each g, h, ℓ_1, ℓ_2, the connected sum $Z^{\ell_1,\ell_2}_{g,h}(m)$ has non-trivial stable cohomotopy Seiberg-Witten invariants by the new non-vanishing theorem in [28]. In particular, X has monopole classes which are give by

$$\pm c_1(X) \pm c_1(Y_m) + \sum_{i=1}^{b_2(N)} \pm E_i, \tag{29}$$

where we set $N := \ell_1(S^1 \times S^3) \# \ell_2 \overline{\mathbb{C}P^2}$ and $E_1, E_2, \cdots, E_k$ is a set of generators for $H^2(N, \mathbb{Z})/\text{torsion}$ relative to which the intersection form is diagonal and the $\pm$ signs are arbitrary and independent of one another.

Then, for each g, h, ℓ_1, ℓ_2, we show that $V := \{Z^{\ell_1,\ell_2}_{g,h}(m)\}_{m \in \mathbb{N}}$ contains infinitely many diffeo types. In fact, suppose that the sequence V contains only finitely many diffeomorphism types. Namely, suppose that there exists a positive integer m_0 such that $Z^{\ell_1,\ell_2}_{g,h}(m_0)$ is diffeomorphic to $Z^{\ell_1,\ell_2}_{g,h}(m)$ for any integer $m \geq m_0$. Then, by taking $m \to \infty$, we see that the set of monopole classes of 4-manifold $Z^{\ell_1,\ell_2}_{g,h}(m_0)$ is unbounded by (29). However, this is a contradiction because the set of monopole classes of any given smooth 4-manifold with $b^+ > 1$ must be finite [27]. Therefore, the sequence V must contain infinitely many diffeomorphism types. Then, we get immediately the desired result by using Theorem C. ∎

3.4 Generalization of the FZZ conjecture

One of the motivations of FZZ conjecture is coming from a result [30] on Einstein 4-manifolds because a typical example of non-singular solution of the normalized Ricci flow is an Einstein metric.

Let X be a closed oriented Riemannian manifold with smooth metric g, and let $\tilde{M}$ be its universal cover with the induced metric $\tilde{g}$. For each $\tilde{x} \in \tilde{M}$, let $V(\tilde{x}, R)$ be the volume of the ball with the center $\tilde{x}$ and radius R. We set

$$\mu(X, g) := \lim_{R \to +\infty} \frac{1}{R} \log V(\tilde{x}, R).$$

Thanks to work of Manning [38], it turns out that this limit exists and is independent of the choice of $\tilde{x}$. We call $\lambda(X, g)$ the volume entropy of the metric g and define the volume entropy of X to be

$$\mu(X) := \inf_{g \in \mathcal{R}^1_X} \lambda(X, g),$$

where $\mathcal{R}^1_X$ means the set of all Riemannian metrics g with unit volume $vol_g = 1$. Then, it is known [31] that any closed Einstein 4-manifold X satisfies

$$2\mathrm{e}(X) - 3|\mathrm{sign}(X)| \geq \frac{1}{54\pi^2}\mu(X)^4. \tag{30}$$

The inequality (5) can be derived from (30) because $n^{n/2}||M|| \geq n!\mu(M)^n$ holds, where n is the dimension of a given manifold M. Hence, the inequality (30) is more stronger than the inequality (5). Based on this result on Einstein case, it is natural to propose the following conjecture which includes Conjecture 4 as a special case:

Conjecture 26 *Let X be a closed oriented smooth Riemannian 4-manifold with $\mu(X) \neq 0$ and $\bar{\lambda}(X) < 0$. Suppose that there is a non-singular solution to the normalized Ricci flow on X. Then the following holds:*

$$2e(X) - 3|sign(X)| \geq \frac{1}{54\pi^2}\mu(X)^4. \tag{31}$$

In the following, we shall show that the converse of this conjecture also does not hold in general. In [15], Gompf showed that, for arbitrary integers $\alpha \geq 2$ and $\beta \geq 0$, one can construct a simply connected symplectic spin 4-manifold $X_{\alpha,\beta}$ satisfying

$$\Big(\mathrm{e}(X_{\alpha,\beta}), \mathrm{sign}(X_{\alpha,\beta})\Big) = \Big(24\alpha + 4\beta, -16\alpha\Big). \tag{32}$$

Notice also that this implies that $b^+(X_{\alpha,\beta}) = 4\alpha + 2\beta - 1$, $2\mathrm{e}(X_{\alpha,\beta}) + 3\mathrm{sign}(X_{\alpha,\beta}) = 8\beta$ and $2\mathrm{e}(X_{\alpha,\beta}) - 3\mathrm{sign}(X_{\alpha,\beta}) = 8(12\alpha + \beta)$. In the following, we shall call $X_{\alpha,\beta}$ the Gompf manifold of degree (α, β). We have $b^+(X_{\alpha,\beta}) \equiv 3 \pmod 4$ if $4\alpha + 2\beta - 1 \equiv 3 \pmod 4$ is satisfied. The Gompf manifold $X_{\alpha,\beta}$ is simply connected, we get $b_1(X_{\alpha,\beta}) = 0$. In particular, $X_{\alpha,\beta}$ is BF-admissible in the case where $4\alpha + 2\beta - 1 \equiv 3 \pmod 4$.

Lemma 27 *Let X be a closed oriented smooth 4-manifold and consider the following connected sum*

$$M := X \# X_{\alpha,\beta} \# (\Sigma_h \times \Sigma_g) \# \ell_1 (S^1 \times S^3) \# \ell_2 \overline{\mathbb{CP}^2}.$$

Then, there are infinitely many integers $\alpha, \beta, g, h, \ell_1, \ell_2$ *for which the following conditions are satisfied simultaneously:*

$$4\alpha + 2\beta - 1 \equiv 3 \pmod 4, \tag{33}$$

$$2e(X) - 3sign(X) + 8(12\alpha + \beta) > (\frac{128}{27} - 4)(g-1)(h-1) + 4(2+\ell_1) - 5\ell_2, \tag{34}$$

$$2e(X) + 3sign(X) + 8\beta > (\frac{128}{27} - 4)(g-1)(h-1) + 4(2+\ell_1) + \ell_2, \tag{35}$$

$$4(2+\ell_1) + \ell_2 > \frac{1}{3}\Big(2e(X) + 3sign(X) + 8\beta + 4(1-h)(1-g)\Big). \tag{36}$$

Proof. First of all, notice that the inequality (34) is always satisfied by taking sufficiently large β for any fixed $\alpha, \ell_1, \ell_2, g, h$. And notice also that there are infinitely many integers α, β, for which (33) is satisfied.

On the other hand, the inequality (35) is equivalent to

$$c + 8\beta - (\frac{128}{27} - 4)(g-1)(h-1) > 4(2+\ell_1) + \ell_2. \tag{37}$$

where $c := 2\mathrm{e}(X) + 3\mathrm{sign}(X)$. Therefore, by (36) and (37), it is enough to prove that there exist infinitely many positive integers $\alpha, \beta, \ell_1, \ell_2, g, h$ satisfying $D > 4(2+\ell_1) + \ell_2 > E$, where we set as

$$D := c + 8\beta - (\frac{128}{27} - 4)(g-1)(h-1), \;\; E := \frac{1}{3}\Big(c + 8\beta + 4(1-h)(1-g)\Big).$$

Notice that both D and E can become sufficiently large positive integers by taking sufficiently large β. We also have

$$D - E = \frac{2}{3}c + \frac{16}{3}\beta - (\frac{128}{27} + \frac{4}{3} - 4)(g-1)(h-1).$$

From this, we see that $D - E$ can become a large positive integer by taking large β. Since there are infinitely many such a β, we are able to conclude that there are also infinitely many ℓ_1, ℓ_2 satisfying $D > 4(2+\ell_1) + \ell_2 > E$. From these observations, we are able to obtain the desired result. ∎

A connected closed manifold X of dimension n is called essential [17] if there exists a map $X \to K$ to an aspherical complex K that does not contract to the $(n-1)$-skeleton of K. It is known that every simply connected manifold is nonessential. Furthermore, a product of arbitrary manifolds with simply connected manifolds is also nonessential. And it is also known that any nonessential manifold has zero volume entropy. Let X and Y be two connected closed oriented manifolds. If Y is nonessential, then it is proved in [9] that

$$\mu(X\#Y) = \mu(X) \tag{38}$$

By Lemma 27 and (38), we get:

Proposition 28 *Let X_m be a nonessential closed oriented smooth 4-manifold and consider the following connected sum*

$$M := X\#X_{\alpha,\beta}\#(\Sigma_h \times \Sigma_g)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{C}P^2}.$$

Then, there are infinitely many integers $\alpha, \beta, g, h, \ell_1, \ell_2$ for which the following conditions are satisfied simultaneously:

$$4\alpha + 2\beta - 1 \equiv 3 \pmod 4, \tag{39}$$

$$2e(M) - 3|sign(M)| > \frac{1}{54\pi^2}\mu(M)^4 \neq 0, \tag{40}$$

$$4(2+\ell_1) + \ell_2 > \frac{1}{3}\Big(2e(X) + 3sign(X) + 8\beta + 4(1-h)(1-g)\Big). \tag{41}$$

Proof. First of all, notice that $\overline{\mathbb{C}P^2}$ and $S^1 \times S^3$ is nonessential (see also [9]). X is also nonessential by the assumption. By (38), we have $\mu(M) = \mu(\Sigma_h \times \Sigma_g)$. Moreover, Corollary 2.2 in [9] tells us that we also have $16(g-1)(h-1) \leq \mu(\Sigma_h \times \Sigma_g)^4 \leq 256\pi^2(g-1)(h-1)$. Therefore, we obtain

$$\frac{16}{54\pi^2}(g-1)(h-1) \leq \frac{1}{54\pi^2}\mu(M)^4 \leq \frac{127}{27}(g-1)(h-1) \tag{42}$$

This particularly tells us that $\mu(M)^4 \neq 0$ whenever $g, h \geq 2$. On the other hand, notice that (41) is nothing but (36). Moreover, by Lemma 22, we have $2e(M) + 3sign(M) = 2e(X) + 3sign(X) + 8\beta + 4(g-1)(h-1) - 4(2+\ell_1) - \ell_2$. Therefore, the inequality (35) is nothing but

$$2e(M) + 3sign(M) > \frac{1}{54\pi^2}\mu(M)^4. \tag{43}$$

Similarly, since Lemma 22 also tells us that $2e(M) - 3sign(M) = 2e(X) - 3sign(X) + 8(12\alpha + \beta) + 4(g-1)(h-1) - 4(2+\ell_1) + 5\ell_2$, the inequality (34) is equivalent to

$$2e(M) - 3sign(M) > \frac{1}{54\pi^2}\mu(M)^4. \tag{44}$$

By (43) and (44), we obtain (40). ∎

Finally, we obtain the following result:

Theorem D *Let X be a BF-admissible, nonessential closed oriented smooth 4-manifold, $X_{\alpha,\beta}$ is the Gompf manifold with degree (α, β) and consider the following connected sum:*

$$M := X\#X_{\alpha,\beta}\#(\Sigma_h \times \Sigma_g)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{C}P^2}$$

where $\ell_1, \ell_2 \geq 1$, and $g, h \geq 3$ are odd integers. And $\alpha \geq 2$ and $\beta \geq 0$. Then, there are infinitely many integers $\alpha, \beta, g, h, \ell_1, \ell_2$ for which M has the following properties.

1. *M has $\mu(M) \neq 0$ and satisfies the strict case of the inequality (31):*

$$2e(M) - 3|sign(M)| > \frac{1}{54\pi^2}\mu(M)^4.$$

2. *M admits at least one smooth structure for which no for which Perelman's $\bar{\lambda}$ invariant is negative and there is no quasi-non-singular solution to the normalized Ricci flow for any initial metric.*

Proof. By Proposition 28, there are infinitely many integers $\alpha, \beta, g, h, \ell_1, \ell_2$ for which (39), (40) and (41) hold. Notice that $X_{\alpha,\beta}$ is BF-admissible under (39). Since X is also BF-admissible, under (41), there is no non-singular solution to the normalized Ricci flow on M for any initial metric by Corollary 21. Moreover, we also obtain $\bar{\lambda}(M) < 0$ by Corollary 14. ∎

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