UNIVERSITY OF MASSACHUSETTS
DEPARTMENT OF MATHEMATICS AND STATISTICS
ADVANCED EXAM - STATISTICS (II)
Tuesday, January 15, 2019

Work all problems and show all work. Explain your answers. State the theorems used whenever possible. 70 points are required to pass.

1. Let $\{X_n\}_{n\geq 1}$ be a sequence of real-valued random variables and X be another real-valued random variable. Suppose that X_n has distribution function $F_n(x)$ for each n and X has distribution function $F(x)$.

   (a) State the definition of almost sure convergence or convergence with probability 1 (denoted as $X_n \xrightarrow{a.s.} X$).

   (b) State the definition of convergence in probability (denoted as $X_n \xrightarrow{P} X$).

   (c) State the definition of convergence in a-th mean (denoted as $X_n \xrightarrow{a} X$).

   (d) State the definition of convergence in distribution (denoted as $X_n \xrightarrow{d} X$).

2. Suppose that X_n is distributed as a Bernoulli(p_n) where $n = 1, 2, \ldots$. That is, $P(X_n = 1) = 1 - P(X_n = 0) = p_n$.

   (a) Suppose that $p_n = 1/n^2$. Prove that $X_n \xrightarrow{a.s.} 0$. You may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

   (b) Suppose that $p_n = 1/n^2$. Suppose that $F_n(x)$ is the distribution function of X_n and $F(x)$ is the distribution function of a constant zero random variable (i.e., $X = 0$). Prove that $F_n(x) \to F(x)$ as $n \to \infty$ for all real numbers x (i.e., $X_n \xrightarrow{d} 0$ and $F_n(0) \to F(0)$ as $n \to \infty$).

   (c) Suppose that $p_n = 1/n$. Prove that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} 0$. You may use the fact that $\sum_{i=1}^{n} \frac{1}{i}$ is asymptotically equivalent to $\log n$, denoted as $\sum_{i=1}^{n} \frac{1}{i} \sim \log n$.

3. Let $X_1, \ldots, X_n$ be a sequence of independent random variables such that $X_n = \sqrt{n}$ with probability $1/2$ and $X_n = -\sqrt{n}$ with probability $1/2$, for $n = 1, 2, \ldots$.
   Find the asymptotic distribution of $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$.
   (Hint) : Check the Lindeberg condition.

4. Suppose that $Y_1, Y_2, \cdots$ are independent and identically distributed from a distribution with density function $f_\theta(y) = \theta/y^{\theta+1}$, $y > 1$ and $\theta > 2$. Note that $E(Y_i) = \frac{\theta}{\theta-1}$ and $Var(Y_i) = \frac{\theta}{(\theta-2)(\theta-1)^2}$.

   (a) Find the asymptotic distribution of $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$.

   (b) Consider an estimator $\tilde{\theta}_n = \frac{\bar{Y}_n}{\bar{Y}_n - 1}$. Find the asymptotic distribution of $\tilde{\theta}_n$.

   (c) Find the root of the likelihood equation, and show that the likelihood equation has a unique solution, denoted as $\hat{\theta}_n$.

   (d) Check the regularity conditions necessary for consistency of the root of the likelihood equation.

   (e) Find the asymptotic distribution of $\hat{\theta}_n$ in (c).

   (f) Is $\tilde{\theta}_n$ in (b) asymptotically efficient? Justify your answer.

5. Suppose that our problem is to find $\mu = E_p(f(X)) = \int_{\mathcal{D}} f(x)p(x)dx$ where p is a probability density function on $\mathcal{D} \subset \mathcal{R}$ (real line) and f is the integrand. The **importance sampling** technique for approximating μ is to sample from an **importance distribution** q, that is a positive probability density function on $\mathcal{R}$ and use $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n f(Y_i)p(Y_i)/q(Y_i)$ to approximate μ. Here n is the sample size and Y_i is a random (i.i.d) sample drawn from the importance distribution q (i.e., $Y_1, \ldots, Y_n \overset{i.i.d}{\sim} q$).

   (a) Show that $\hat{\mu} \xrightarrow{P} \mu$ when n goes to infinity.

   (b) Suppose that p is a uniform distribution on (0,1), denoted by $U(0,1)$, q is a uniform distribution on (0,1/2), denoted by $U(0,1/2)$, and that $f(x) = x^2$. Show that $\hat{\mu} \xrightarrow{P} E_q(f(Y)p(Y)/q(Y))$ which is not equal to $E_p(f(X))$. In this case, is q an appropriate importance distribution for approximating $E_p(f(X))$?

   (c) Suppose that $q(x) = \exp(-x)$ for $x > 0$ and that $f(x) = |x|$ for all $x \in \mathcal{R}$, so that $q(x) = 0$ at some x where $f(x) \neq 0$. Give a density $p(x)$ for which the expectation of $f(X)p(X)/q(X)$ for $X \sim q$ matches the expectation of $f(X)$ for $X \sim p$.