UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Thursday, January 18, 2018

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each answer is worth approximately the same number of points.

1. Let (X_i, Y_i), $i = 1, \ldots, n$ be a sequence of i.i.d. random vectors, where X_i is an exponential distribution with the density function (denoted as $X_i \sim \text{exponential}(\theta)$)

$$f(x \mid \theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right)$$

with $x > 0$, $\theta > 0$ and the mean $E(X_i) = \theta$, and $Y_i \sim \text{exponential}(1/\theta)$. Consider

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

   (a) Find the value of c minimizing $E(\hat{\theta} - \theta)^2$ where $\hat{\theta} = c\bar{X}_n$ and c is a constant.

   (b) Show that $\bar{X}_n$ is consistent for θ and find the asymptotic distribution (including asymptotic variance) of $\bar{X}_n$.

   (c) Show that $\frac{1}{\bar{Y}_n}$ is consistent for θ and find the asymptotic distribution (including asymptotic variance) of $\frac{1}{\bar{Y}_n}$.

2. Let $X_1, \ldots, X_n$ be an iid example from an exponential distribution with the density function $f(x \mid \theta) = \theta e^{-\theta x}$ where $x > 0$ and $\theta > 0$.

   (a) Using $Q(T, \theta) = 2\theta T$ where $T = \sum_{i=1}^{n} X_i$, give the explicit expressions necessary to construct a $100(1-\alpha)\%$ confidence interval for θ based on $Q(T, \theta)$.

   (b) Suppose one takes a prior for θ, $f(\theta) \propto \frac{1}{\theta}$ where $\theta > 0$. Find the posterior distribution for θ, $f(\theta \mid X_1, \ldots, X_n)$.

   (c) Find the Bayes estimator of θ under squared error loss.

   (d) Describe how to construct the following two types of Bayesian posterior credible interval for θ:
   (i) $100(1-\alpha)\%$ Bayesian interval based on quantiles of the posterior distribution,
   (ii) $100(1-\alpha)\%$ Highest Posterior Density(HPD) interval.

3. Let $X_1, \ldots X_n$ be a random sample from a uniform $(\theta, 2\theta)$ where $\theta > 0$.
   (a) Find a minimal sufficient statistic for θ.
   (b) Obtain the method of moments estimator of θ.
   (c) Obtain the MLE of θ.
   (d) Find an approximate $100(1-\alpha)\%$ confidence interval for θ for large n.

4. Suppose μ_i has a normal prior distribution with mean 0 and variance A, while z_i given μ_i is normal with mean μ_i and variance 1,

$$\mu_i \sim \mathcal{N}(0, A), \quad z_i|\mu_i \sim \mathcal{N}(\mu_i, 1),$$

   where (μ_i, z_i) pairs are independent for $i = 1, \ldots, N$. Let $\mathbf{z} = [z_1, \ldots, z_N]$.
   (a) What is the maximum likelihood estimator $\hat{\mu}_i^{\text{MLE}} = t(\mathbf{z})$ given observed $\mathbf{z}$?
   (b) Show that the posterior distribution is $\mu_i|\mathbf{z} \sim \mathcal{N}(Bz_i, B)$ where $B = \frac{A}{A+1}$.
   (c) What is the Bayes estimator $\hat{\mu}_i^{\text{Bayes}} = t(\mathbf{z}) = E[\mu_i|\mathbf{z}]$?
   (d) While the maximum likelihood estimator is unbiased, the Bayes estimator is biased for finite N. What happens to the bias of the Bayes estimator as the sample size $N \to \infty$?