DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
JANUARY 2018

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. A matrix $\mathbf{A}$ is *strictly diagonally dominant* if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad \text{for } i = 1, \cdots, n\,.$$

Prove that if a matrix is strictly diagonally dominant, then no pivoting is necessary for Gaussian elimination. (Hint: Prove that the lower right corner of the partly processed matrix is also diagonally dominant.)

2. Use Newton's method to find one root of the function

$$f(x) = x^3 - (2a+2)x^2 + (a^2+4a)x - 2a^2 = (x-2)(x-a)^2.$$

Suppose the initial guess is sufficiently close to $x = 2$.

(a) For which values of a, Newton's method has only the first order convergence? Compute the convergence rate.

(b) For what values of a, Newton's method has the second order convergence?

3. We want to approximate

$$\int_0^2 f(x)x^2 \mathrm{d}x$$

by a rule of the form $af(b)$. Find a and b so that the method is exact for polynomials of the highest possible degree. Also find the error term.

4. Consider the one-step method to approximate the solution of $y' = f(y)$, $y(t_0) = y_0$:

$$\begin{cases} k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + h, y_n + hk_1) \\ y_{n+1} &= y_n + \frac{1}{2}h(k_1 + k_2) \end{cases}$$

where $h = t_{n+1} - t_n$.

(a) Find a simplified expression for the truncation error of this scheme.

(b) Is the scheme consistent? Explain.

5. Find the values of a and b which solve the following optimization problem:

$$\min_{a,b} \int_0^\infty (e^x - ax - b)^2 e^{-3x} \mathrm{d}x\,.$$

Note that the function $f(x) = (ax+b)$ is the weighted L^2 projection of e^x onto the space spanned by $\{1, x\}$.

6. Define function $f(x)$ as

$$f(x) = \begin{cases} \sin x, & x \in [0,1], \\ \cos x, & x \in (1, 2\pi]. \end{cases}$$

(a) Find the second order polynomial interpolation to $f(x)$, with interpolation points $\{0, \pi, 2\pi\}$. Also compute the maximum error of the above interpolation.

(b) Prove that the maximum error for polynomial interpolation of any degree will be NOT less than $|\sin 1 - \cos 1|/2$.

7. Suppose A is a positive definite matrix $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. For $n \geq 2$ please

(a) Prove $\mathbf{A}$ is non-singular.

(b) Is A a symmetric matrix? If yes, prove it. Otherwise, give a counter example.