NAME:

Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question and on the blank page after each question.

1. Let A be a 3×3 matrix whose entries are real constants. Write down necessary and sufficient conditions on A so that any $x(t)$ solving the first order system

$$\dot{x} = Ax \quad t \in \mathbb{R}$$

remains bounded as $t \to \pm\infty$. Then provide necessary and sufficient condition on A so that any $x(t)$ solving the same system remains bounded as $t \to \infty$ (the behavior as $t \to -\infty$ not necessarily remaining bounded).

2. Consider the two dimensional system

$$\dot{x}_1 = x_1 - x_1^2 + x_2$$
$$\dot{x}_2 = -x_2 + \lambda x_1$$

for some $\lambda \in (0,1)$. Show there is a solution $(x_1(t), x_2(t))$ of the system defined for all $t \in \mathbb{R}$ and such that

$$\lim_{t \to -\infty} x_1(t) = 0, \ \lim_{t \to -\infty} x_2(t) = 0$$

and at the same time

$$\lim_{t \to +\infty} x_1(t) = \lambda + 1, \ \lim_{t \to +\infty} x_2(t) = \lambda(\lambda + 1)$$

3. Consider the following nonlinear ODE

$$\frac{d^2}{dt^2}x + \tfrac{1}{2}(2 - \sin(t)) \left(\frac{d}{dt}x \right)^3 + x = 0.$$

Show that the origin of the phase plane, that is $(x(0), \dot{x}(0)) = (0, 0)$ is an asymptotically stable fixed point for this system.

4. Show that the function

$$E(x,y) = \tfrac{1}{2}y^2 - \cos(x)$$

is non-increasing along all trajectories of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -3y - \sin(x). \end{aligned}$$

Then, sketch the phase diagram for this system in the strip $\{(x,y)\ :\ -\tfrac{3}{2}\pi < x < \tfrac{3}{2}\pi\}$. Indicate fixed points (with their classification), periodic orbits, and connecting orbits.

5. Use Duhamel's principle and d'Alembert's formula to solve the initial value problem in the line

$$\partial_{tt}u - \partial_{xx}u = \sin(x),$$
$$u(x,0) = \sin(x),\ \partial_t u(x,0) = \cos(x).$$

6. Let $u(x,t)$ solve for $x \in \mathbb{R}$ and $t \geq 0$ the following wave equation with damping terms

$$\partial_{tt}u + \alpha\partial_t u - \partial_{xx}u - \beta\partial_{xxt}u = 0.$$

Here α, β are two given positive constants. Assume that for every fixed $t \geq 0$, the function $u(x,t)$ is of rapid decay at infinity with respect to t.

(a) Let

$$E(t) := \frac{1}{2}\int_{\mathbb{R}} (\partial_t u(x,t))^2 + (\partial_x u(x,t))^2 \, dx.$$

Show that $\dot{E}(t) \geq 0$.

(b) Show that the only possible solution of the equation which vanishes rapidly at infinity in space for every fixed t and such that $u(x,0) \equiv \partial_t u(x,0) \equiv 0$ is the zero function.

(c) Use the previous two steps to prove uniqueness for this equation (among functions with rapid decay at infinity in space).

7. Let $u(x, y)$ be a harmonic function in some bounded region $\Omega \subset \mathbb{R}^2$ with smooth boundary. Assume that u has derivatives which are continuous uniformly up to $\partial\Omega$. Let $v = |\nabla u|^2$.

   (a) Show that v is a subharmonic function.

   (b) Use the previous part to show the estimate

$$\sup_{\Omega} |\nabla u| \leq \sup_{\partial\Omega} |\nabla u|.$$

8. Let $u(x, y)$ be a function in $\mathbb{R}^2$ which is periodic with period 1 in each variable, and solving Poisson's equation

$$\partial_{xx}u(x, y) + \partial_{yy}u(x, y) = f(x, y),$$

for some given function $f \in C^0(\mathbb{R}^2)$. Show that

$$\int_{[0,1]^2} f(x, y)\, dxdy = 0.$$