Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

Tuesday, January 17, 2017

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Show that for any subset $A \subset \mathbb{R}^n$, the quotient space $\mathbb{R}^n/A$ obtained by identifying A to a point is Hausdorff if and only if A is closed.

2. Let $A \subset \mathbb{R}^2$ be countable. Prove that $\mathbb{R}^2 \setminus A$ is connected.

3. Let $\mathbb{R}^\omega$ be the set of all real-valued sequences, and let $B \subset \mathbb{R}^\omega$ be the subset of all bounded sequences.

   (a) If $\mathbb{R}^\omega$ is given the product topology, is B open, closed, both or neither?

   (b) Answer the same question if $\mathbb{R}^\omega$ is given the box topology.

4. Let $C_1 \supset C_2 \supset C_3 \supset \ldots$ be a nested sequence of nonempty compact subsets of a Hausdorff space X. Show that the intersection $\bigcap_i C_i$ is nonempty and compact.

5. For any $n > 0$, let $\sim$ be the equivalence relation on $\mathbb{R}^n$ given by $x \sim y$ if and only if there exists $t > 0$ so that $x = ty$. Let $X = \mathbb{R}^n/\sim$ be the associated quotient space, and let $q\colon \mathbb{R}^n \to \mathbb{R}^n/\sim$ be the quotient map.

   (a) Show that X is not Hausdorff.

   (b) Show that $X \setminus q(0)$ is homeomorphic to S^{n-1}.

   (c) Is the map
   $$f\colon \mathbb{R}^n \to \mathbb{R} \times X,\ f(x) = (|x|, q(x))$$
   an embedding, i.e. a homeomorphism onto its image? Prove your answer.

6. Compute the fundamental groups of $\mathbb{R}^3 \setminus (C \cup L)$ and of $\mathbb{R}^3 \setminus C$, where C is the circle $x^2 + y^2 = 1$ in the plane $z = 0$ and L is the line $x = y = 0$.

7. Show that the composition $g \circ f$ of any two continuous maps $f\colon S^1 \to S^2$ and $g\colon S^2 \to S^1$ is homotopic to a constant map. Is the same true if S^2 is replaced by the torus $T^2 = S^1 \times S^1$? Explain your answer.