DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - STATISTICS
January 20, 2017

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let $X_1, \dots X_n$ be a random sample from a gamma distribution with probability density function:

$$f(x|\alpha,\beta) = \frac{x^{\alpha-1}\exp(-x/\beta)}{\Gamma(\alpha)\beta^{\alpha}}, x > 0, \alpha > 0, \beta > 0.$$

Consider the prior distribution for the parameter β as an inverse gamma distribution with the following probability density function [i.e., $\beta \sim$ Inverse Gamma(λ,θ)]:

$$p(\beta|\lambda,\theta) = \frac{\exp(-1/(\theta\beta))}{\Gamma(\lambda)\theta^{\lambda}\beta^{\lambda+1}}, \beta > 0, \lambda > 0, \theta > 0.$$

Note that for $\beta \sim$ Inverse Gamma(λ,θ), the following are known:

$$E(\beta) = \left[\frac{1}{\theta(\lambda-1)}\right], \text{ for } \lambda > 1,$$
$$Var(\beta) = \left[\frac{1}{\theta^2(\lambda-1)^2(\lambda-2)}\right], \text{ for } \lambda > 2.$$

You may assume $\alpha, \theta,$ and λ are known.

(a) (6 points) Find the posterior distribution of β.

(b) (6 points) Find the posterior mean of β.

(c) (6 points) Describe how to construct a 95% equal-tail posterior credible interval for β.

2. Suppose $X_1 \dots X_n \sim^{iid}$ Poisson(λ).

(a) (5 points) Give or find the MLE, $\hat{\lambda}$, of λ.

(b) (5 points) Find the information, $I_n(\lambda)$.

(c) (5 points) Find the asymptotic distribution of the MLE.

(d) (5 points) Find a $1-\alpha$ asymptotic confidence interval for λ (hint: the endpoints of the interval may not include parameters).

Now suppose $X_1, \dots X_{n_1} \sim^{iid}$ Poisson$(\lambda_1), Y_1, \dots Y_{n_2} \sim^{iid}$ Poisson(λ_2). We wish to test the hypothesis $H_0 : \lambda_1 = \lambda_2$ against the alternative $H_1 : \lambda_1 \neq \lambda_2$.

(e) (6 points) For large n_1 and n_2, find the approximate distribution of the difference in MLEs: $\hat{\lambda}_1 - \hat{\lambda}_2$.

(f) (6 points) Show that under the null hypothesis, the test statistic

$$T = \frac{(\hat{\lambda}_1 - \hat{\lambda}_2)^2}{(\frac{1}{n_1} + \frac{1}{n_2})\hat{\lambda}},$$

where $\hat{\lambda} = \frac{\sum X_i + \sum Y_i}{n_1+n_2}$, approaches a χ^2 distribution as $n_1, n_2 \to \infty$. Find the degrees of freedom of the χ^2 distribution.

3. Consider $X_1, X_2, \dots X_n \sim$ iid $U(0,\theta)$, with parameter θ unknown.

(a) (6 points) Show that the maximum likelihood estimator $\hat{\theta}$ of θ is $X^{(n)}$, the maximum of the observations.

(b) (6 points) If possible, find the exact distribution (pdf or cdf) of $\hat{\theta}$. If not possible, find its asymptotic distribution.

(c) (6 points) Is $\hat{\theta}$ consistent for θ? Show.

4. (8 points) Consider the following procedure for testing the hypothesis $H_0 : p \geq 0.5$ against the alternative $H_1 : P < 0.5$ in a binomial distribution with parameters $n = 10$ and p. We take observation X_1 and reject H_0 if $X_1 = 0$ or do not reject if $X_1 \geq 9$. If $X_1 \in \{1, 2, 3, \dots 8\}$, we take another observation X_2 and reject if $X_1 + X_2 < 5$, do not reject if $X_1 + X_2 \geq 5$. Determine the power function of this procedure. You may use the notation $P(x|n,p)$ to represent the probability that a binomial random variable with parameters n and p takes the value x.

5. $X_1, X_2, \ldots$ is an infinite sequence of i.i.d. Normally distributed random variables with mean μ and standard deviation σ. We wish to estimate μ. *consistent* and *asymptotically efficient* are two terms that can be applied to sequences of estimators. Which of them apply to each of the sequences of estimators of μ below? Why?

   (a) (6 points) $\hat{\mu}_n = (X_1 + X_2 + \cdots + X_n)/(n)$.

   (b) (6 points) $\hat{\mu}_n = (X_1 + X_3 + \cdots + X_{n-1})/(n/2)$ for $n = 2, 4, 6, \ldots$.

   For the next two parts, also assume $U_1, U_2, \ldots$ is an infinite sequence of i.i.d. Uniform$(-1, 1)$ random variables independent of all the X_i's.

   (c) (6 points) $\hat{\mu}_n = (\sum_{i=1}^{n} X_i)/(n + U_n)$.

   (d) (6 points) $\hat{\mu}_n = (\frac{1}{n} \sum_{i=1}^{n} X_i) + U_n$.