UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Tuesday, January 19, 2017

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each question is worth 25 points.

1. Let X_1 be Uniform(0,1), and define $X_2|X_1 = x_1$ to be Uniform(0,x_1).
   (a) Find the mean and variance of X_2. For full credit, do not use the pdf of X_2. (Note that you may use that the mean and the variance of Uniform(a,b) are $(a+b)/2$ and $(b-a)^2/12$ respectively.)
   (b) Find the marginal density of X_2.
   (c) Verify that your result from part (b) is a density.

2. Problems related to the Central Limit Theorem.
   (a) State the Central Limit Theorem.
   (b) In theory, the Central Limit Theorem is about a limit as $n \to \infty$. But in practice it is used for finite values of n, as long as n is large enough, to approximate the distribution of $\bar{X}$. Assuming n is large enough, state the approximation.
   (c) By multiplying by n, the Central Limit Theorem also gives an approximation to $X_{sum} \equiv \sum_1^n X_i$. Assuming n is large enough, state the approximation.
   (d) Find the moment generating function of the Normal distribution.

3. Let $X_1, \cdots, X_n$ be iid random variables with mean μ, variance σ^2 (> 0), $\mu_4 = E(X_1 - \mu)^4$, and we assume that $0 < \mu_4 < \infty$ as well as $\mu_4 > \sigma^4$. Denote $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $S_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ for $n \geq 2$.
   (a) Define $W_n = (n-1)n^{-1}S_n^2$ and $Y_i = (X_i - \mu)^2$, $i = 1, \cdots, n$, and $\bar{Y}_n = n^{-1}\sum_{i=1}^n Y_i$. Show that
   $$W_n = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \bar{Y}_n - (\bar{X}_n - \mu)^2.$$
   (b) Let $U_n = \sqrt{n}(\bar{Y}_n - \sigma^2)$. Show that U_n converges in distribution to $N(0, \mu_4 - \sigma^4)$.
   (c) Let $V_n = \sqrt{n}(\bar{X}_n - \mu)^2$. Show that V_n converges to zero in probability.
   (d) Show that $\sqrt{n}(W_n - \sigma^2) = U_n + V_n$ and
   $$\sqrt{n}(S_n^2 - \sigma^2) = \sqrt{n}(\frac{n}{n-1}W_n - \sigma^2) = \sqrt{n}(W_n - \sigma^2) - \frac{\sqrt{n}}{n-1}W_n$$
   and therefore $\sqrt{n}(S_n^2 - \sigma^2)$ converges in distribution to $N(0, \mu_4 - \sigma^4)$, which is the Central Limit Theorem for the sample variance.

4. Granite, like any other rock, can contain naturally occurring radioactive elements like radium, uranium and thorium. Typically, however, granite has a very low rate of radioactivity. This question concerns a piece of granite that releases radioactive particles according to a Poisson process with a rate of one particle per hour. Let X be the number of particles released in a particular hour.

   (a) Write down the pdf of X.

   (b) Find the moment generating function of X.

   (c) The particular hour we're studying can be represented by the unit interval $(0, 1]$, which can be partitioned into the subintervals $(0, \frac{1}{2}], (\frac{1}{2}, 1]$. Let X_1^2 and X_2^2 be the numbers of particles released in the first and second subintervals, respectively, so that $X = X_1^2 + X_2^2$. The superscript 2 indicates that the interval is divided into 2 pieces.

      i. What is the distribution of X_i^2, for $i = 1, 2$?

      ii. Can moment generating functions be used to show $X = X_1^2 + X_2^2$? Why or why not?