COMPLEX ANALYSIS BASIC EXAM
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
JANUARY 2017

- Each problem is worth 10 points.
- Passing Standard: **Do 8 of the following 10 problems**, *and*
  - Master's level: 45 points with three questions essentially complete
  - Ph. D. level: 55 points with four questions essentially complete

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1. Show that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1} = \frac{\pi}{\sin \pi a} \quad \text{for } 0 < a < 1.$$
Show the contour and prove all estimates you use.

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2. Let $f(z)$ be analytic inside and on the circle C: $|z| = R > 0$. Suppose $|f(z)| < M$ along C.

(a) For $n \geq 1$, give the precise statement of the Cauchy inequality for the n-th derivative $f^{(n)}(x)$ of f.

(b) Given examples to show that Cauchy inequality is the best possible, i.e. the estimate growth of $f^{(n)}(x)$ cannot be improved for all functions analytic inside and on C. *Justify your reasoning.*

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3. Let f be a holomorphic function on the open unit disk D. Suppose f extends continuously to the closed unit disk $\overline{D}$, and supppose f vanishes on the semicircle $z = e^{i\theta}$ where $0 \leq \theta \leq \pi$. Show that $f = 0$ on $\overline{D}$. *Justify your reasoning.*

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4. Recall that a function $g(z)$ is said to be have a pole of order $m \geq 0$ at infinity if $g(1/z)$ has a pole of order m at the origin. Find all meromorphic functions $f(z)$ on the extended complex plane that have a pole of order m at the origin, a pole order n at infinty, and nowhere else. *Justify your reasoning.*

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5. Determine all singularities of the function
$$f(z) = \frac{(z-1)^2(z+3)}{1-\sin(\pi z/2)}$$
and determine for each one its type.

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6. Find a conformal mapping φ that takes the interior of the unit disk to the open set $W := \{|w-1| < 1\}$, such that $\varphi(0) = 1/2$ and $\varphi(1) = 0$. Is φ uniquely determined? *Justify your reasoning.*

Note: Of course the point 1 is *not* in the *interior* of the unit disk.

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7. Let U be a. non-empty open set of the complex plane, For $z \in U$, write $z = x + iy$. Let $f(z) = u(x, y) + iv(x, y)$ be a function defined on U such that its real part $u(x, y)$ and it imaginary part $v(x, y)$ are both real analytic functions in the two real variables x, y. Define

$$\frac{\partial f}{\partial z} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \quad \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

Show that f is analytic if and only if $\frac{\partial f}{\partial \overline{z}} = 0$, in which case show that $f' = \frac{\partial f}{\partial z}$. *Justify your reasoning.*

Note: This is a standrad result; please prove it from first principles.

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8. Let $f : \mathbf{C} \to \mathbf{C}$ be an analytic function. Let ω_1, ω_2 be complex numbers which are linearly independent over $\mathbf{R}$. Suppose

$$f(z) = f(z + \omega_1) = f(z + \omega_2)$$

for all $z \in \mathbf{C}$. Show that f is constant. *Justify your reasoning.*

Note: This is a standrad property about *elliptic functions*. Do **not** cite this theorem; instead solve this problem using basic results about complex analytic functions.

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9. Let U be a non-empty, connected open subset of $\mathbf{C}$. Fix a point $\alpha \in U$, and let α_n be a sequence of points in $U \backslash \{\alpha\}$ that converges to α. Let f, g be holomorphic functions on U that do no vanish at any point of U. Show that if

$$\frac{f'(\alpha_n)}{f(\alpha_n)} = \frac{g'(\alpha_n)}{g(\alpha_n)}$$

for every α_n, then g is a multiple of f. *Justify your reasoning.*

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10. Fix $\alpha \in \mathbf{C}$, and set $U_\alpha(R) := \{z \in \mathbf{C} : 0 < |z - \alpha| < R\}$. For any analytic function $f : U_\alpha(R) \to \mathbf{C}$, show that the following conditions are equivalent:

- f has a pole of order $k \geq 0$ at α;
- there exists an open disk D_α centered at α, and an anlytic function h on D_α, such that $f(z) = h(z)/(z - \alpha)^k$ for all $z \in D_\alpha \cap U_\alpha$;
- there exists positive real numbers M_1, M_2 such that for all z in a punctured open neighborhood of α, we have $M_1|z - \alpha|^{-k} \leq |f(z)| \leq M_2|z - \alpha|^k$.
