UMass Amherst Algebra Advanced Exam

Wednesday 1/18/17, 10:00AM–1:00PM, LGRT 219.

Instructions: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

Q1. Let $G$ be a non-abelian group of order 28 containing an element of order 4. Describe $G$ in terms of generators and relations.

Q2. Determine the character table of the alternating group $A_4$.

Q3. Let $G$ be a finite group of order $n$ with $m$ conjugacy classes $C_1, \ldots, C_m$. For every $i = 1, \ldots, m$, let $l_i$ be the order of the centralizer $C_G(x_i)$ for $x_i \in C_i$. We can assume that $l_1 \geq \ldots \geq l_m$.

(a) Show that $1 = \frac{1}{l_1} + \ldots + \frac{1}{l_m}$.

(b) Show that $l_m \leq m$, $l_{m-1} \leq 2(m - 1)$, and in fact, for every $p$, $l_p \leq q_p$ for some $q_p$ which depends only on $m$ (and not on $n$).

(c) Show that for every $m \geq 1$ there exists $n \geq 1$ such that every finite group of order larger than $n$ contains more than $m$ conjugacy classes.

2. Commutative Algebra

Q4.

(a) Let $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Prove that $R$ is a unique factorization domain.

(b) Let $S = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Prove that $S$ is not a unique factorization domain.

Q5.

(a) Let $\mathfrak{m} \subset \mathbb{C}[x, y]$ be some maximal ideal in the polynomial ring in two variables over complex numbers. Show that

$$\dim_{\mathbb{C}}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = n + 1$$

for every $n \geq 1$. Here $\mathfrak{m}^n$ is the $n$-th power of the ideal $\mathfrak{m}$.

(b) Let $R = \mathbb{C}[x, y, z]/(xy - z^3)$, the quotient-ring. Show that $R$ is not isomorphic to $\mathbb{C}[x, y]$. 

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Q6. Let \( p \) be a prime ideal of a Noetherian commutative ring \( R \). Let \( M \) and \( N \) be finitely generated \( R \)-modules. Construct a natural isomorphism of the following \( R_p \)-modules:

\[
\text{Hom}_{R_p}(M_p, N_p) \simeq (\text{Hom}_R(M, N))_p
\]

3. Field theory and Galois theory

Q7. Let \( f \in \mathbb{Q}[x] \) be an irreducible cubic polynomial and suppose that \( f \) has a root \( \alpha \in \mathbb{C} \setminus \mathbb{R} \). Let \( K \) be the splitting field of \( f \). Determine the Galois group of \( K \) over \( \mathbb{Q} \).

Q8. Let \( K/k \) be an algebraic extension. Let \( \alpha, \beta \in K \) be different roots of the same irreducible polynomial \( f(x) \in k[x] \) of degree \( n \).

(a) Show that \( \deg_k(\alpha + \beta) \leq \frac{n(n-1)}{2} \). Recall that the degree of an element of the field extension is, by definition, the degree of its minimal polynomial.

(b) For every \( n \geq 1 \), show that there exist fields \( K/k \) and elements \( \alpha, \beta \in K \) as above such that \( \deg_k(\alpha + \beta) = \frac{n(n-1)}{2} \).

Q9.

(a) Let \( p \) be an arbitrary prime number. The Dirichlet theorem on primes in arithmetic progressions implies that there exists a prime number \( q \) such that \( q \equiv 1 \mod p \). Use this to show existence of a Galois extension \( K/\mathbb{Q} \) with Galois group of order \( p \).

(b) Find explicitly an algebraic integer \( \alpha \in \mathbb{C} \) such that the splitting field of its minimal polynomial over \( \mathbb{Q} \) has degree 5.