UMass Amherst Algebra Advanced Exam

Wednesday 1/18/17, 10:00AM–1:00PM, LGRT 219.

Instructions: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

Q1. Let G be a non-abelian group of order 28 containing an element of order 4. Describe G in terms of generators and relations.

Q2. Determine the character table of the alternating group A_4.

Q3. Let G be a finite group of order n with m conjugacy classes $C_1, \dots, C_m$. For every $i = 1, \dots, m$, let l_i be the order of the centralizer $C_G(x_i)$ for $x_i \in C_i$. We can assume that $l_1 \geq \dots \geq l_m$.

(a) Show that $1 = \frac{1}{l_1} + \dots + \frac{1}{l_m}$.

(b) Show that $l_m \leq m$, $l_{m-1} \leq 2(m-1)$, and in fact, for every p, $l_p \leq q_p$ for some q_p which depends only on m (and not on n).

(c) Show that for every $m \geq 1$ there exists $n \geq 1$ such that every finite group of order larger than n contains more than m conjugacy classes.

2. Commutative Algebra

Q4.

(a) Let $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Prove that R is a unique factorization domain.

(b) Let $S = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Prove that S is *not* a unique factorization domain.

Q5.

(a) Let $\mathfrak{m} \subset \mathbb{C}[x, y]$ be some maximal ideal in the polynomial ring in two variables over complex numbers. Show that

$$\dim_{\mathbb{C}}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = n + 1$$

for every $n \geq 1$. Here $\mathfrak{m}^n$ is the n-th power of the ideal $\mathfrak{m}$.

(b) Let $R = \mathbb{C}[x, y, z]/(xy - z^3)$, the quotient-ring. Show that R is not isomorphic to $\mathbb{C}[x, y]$.

Q6. Let $\mathfrak{p}$ be a prime ideal of a Noetherian commutative ring R. Let M and N be finitely generated R-modules. Construct a natural isomorphism of the following $R_{\mathfrak{p}}$-modules:

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \simeq (\mathrm{Hom}_R(M, N))_{\mathfrak{p}}$$

3. Field theory and Galois theory

Q7. Let $f \in \mathbb{Q}[x]$ be an irreducible cubic polynomial and suppose that f has a root $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Let K be the splitting field of f. Determine the Galois group of K over $\mathbb{Q}$.

Q8. Let K/k be an algebraic extension. Let $\alpha, \beta \in K$ be different roots of the same irreducible polynomial $f(x) \in k[x]$ of degree n.

(a) Show that $\deg_k(\alpha + \beta) \leq \frac{n(n-1)}{2}$. Recall that the degree of an element of the field extension is, by definition, the degree of its minimal polynomial.

(b) For every $n \geq 1$, show that there exist fields K/k and elements $\alpha, \beta \in K$ as above such that $\deg_k(\alpha + \beta) = \frac{n(n-1)}{2}$.

Q9.

(a) Let p be an arbitrary prime number. The Dirichlet theorem on primes in arithmetic progressions implies that there exists a prime number q such that $q \equiv 1 \mod p$. Use this to show existence of a Galois extension $K/\mathbb{Q}$ with Galois group of order p.

(b) Find explicitly an algebraic integer $\alpha \in \mathbb{C}$ such that the splitting field of its minimal polynomial over $\mathbb{Q}$ has degree 5.