Department of Mathematics and Statistics
University of Massachusetts Amherst

BASIC EXAM: TOPOLOGY - January 13, 2016

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Masters level, 60% with two questions essentially complete.
For Ph.D. level, 75% with three questions essentially complete.

1. Consider the following topologies on the real line $\mathbb{R}$:
   (i) trivial topology, (ii) discrete topology, (iii) finite complement topology.

   For each topology, determine, with explanations, which one of the following functions from $\mathbb{R} \to \mathbb{R}$ (both the domain and the range taken with the same topology)

   $$f(x) = x^2 \ , \quad g(x) = e^x \ , \quad h(x) = sin(x)$$

   are **(a)** continuous, **(b)** open maps, **(c)** embeddings.

2. For a subset A of a topological space X, let $i(A)$ denote the *interior* of A in X, $\bar{A}$ denote its *closure* in X, and $Bd(A) := \bar{A} \cap \overline{X \setminus A}$ the *boundary* of A.

   (a) Show that $i(A)$ and $Bd(A)$ are disjoint and $\bar{A} = i(A) \cup Bd(A)$.
   (b) Show that $Bd(A) = \emptyset$ if and only if A is both open and closed in X.
   (c) If A is open, is it always true that $A = i(\bar{A})$? Prove or give a counter-example.

3. A space is called *totally disconnected* if the only connected subsets are single point sets.

   (a) Suppose $\{X_\alpha\}_{\alpha \in A}$ is a family of totally disconnected spaces. Show that $X_A := \prod_{\alpha \in A} X_\alpha$ equipped with the product topology, is totally disconnected.
   (b) Let A be a countably infinite set, and let $X_\alpha = \{0, 1\}$ (a 2 point space with the discrete topology) for each $\alpha \in A$. Show that X_A is *not* a discrete space.

4. Let $f : X \to \mathbb{R}$ be a continuous function on a compact metric space. Prove that f is uniformly continuous.

5. Let X be an n-dimensional compact, connected *manifold with boundary* $\partial X \neq \emptyset$. Prove that its boundary $Y = \partial X$ is an $(n-1)$-dimensional manifold.
   Is Y necessarily compact? Connected? Justify your answers.

6. Prove that none of the following spaces are *homeomorphic* to each other:

   $$\mathbb{R}^2, \; S^1 \times \mathbb{R}, \, S^2, \, S^1 \times S^1, \, \mathbb{R}^3, \, S^3.$$

   (Here S^n denotes the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.)

7. Let $\mathbb{RP}^2$ denote the *real projective plane* and $T^2 = S^1 \times S^1$ denote the 2-*torus*. Show that the composition $g \circ f$ of *any* two continuous maps $f : S^1 \to \mathbb{RP}^2$ and $g : \mathbb{RP}^2 \to T^2$ is homotopic to a constant map.