UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Wednesday, January 13, 2016

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose $X_i, i = 1, ..., 100$ are iid Poisson(0.0001).
   (a) (6pts) Find the standard error of the sample mean.
   (b) (7pts) Define $Y = \sum_{i=1}^{100} X_i$. Is $P(Y \geq 1)$ close to zero or close to one? Why? Explain your answer.

2. Suppose that $X_1, \cdots, X_n$ are iid uniformly distributed on the interval $(0, \theta)$
   (a) (6pts) Find the moment-generating function for X_1.
   (b) (7pts) Find the moment-generating function for $V = aX_1 + b$ where $a > 0$ and b are fixed constants. What is the distribution of V? Why? Explain your answer.
   (c) (8pts) Denote $T_n = (\prod_{i=1}^n X_i)^{1/n}$, the geometric mean of $X_1, \cdots, X_n$. Find $c(> 0)$ such that $T_n \xrightarrow{p} c$ as $n \to \infty$.
   (d) (8pts) For the same T_n defined above, find a_n and $b_n(> 0)$ such that $(T_n - a_n)/b_n \xrightarrow{D} N(0, 1)$ as $n \to \infty$.

3. Let $X_1, \cdots, X_n$ be iid with the common pdf given by

$$f(x) = \beta^{-1} \exp(-x/\beta) I(x > 0), \beta > 0.$$

   Deine the order statistics as $Y_1 = X_{(1)}, Y_2 = X_{(2)}, \cdots, Y_n = X_{(n)}$. The X_i's and $X_{(i)}$'s may be interpreted as the failure times and the ordered failure times respectively. Let us denote

$$U_1 = Y_1, U_2 = Y_2 - Y_1, U_3 = Y_3 - Y_2, \cdots, U_n = Y_n - Y_{n-1}$$

   and these are referred to as the spacings between the successive order statistics or failure times.
   (a) (6pts) Find the joint pdf of $Y_1, \cdots, Y_n$. *Hint: the joint pdf of the order statistics $Y_1 = X_{(1)}, Y_2 = X_{(2)}, \cdots, Y_n = X_{(n)}$ is given by $n! f(y_1) \cdots f(y_n)$ for $y_1 < y_2 < \cdots < y_n$, where $f(\cdot)$ is the pdf of X_i.*
   (b) (7pts) Find the joint pdf of $U_1, \cdots, U_n$.
   (c) (8pts) Show that $U_1, \cdots, U_n$ are independently distributed. Find the distribution of each U_i.
   (d) (7pts) Express $Y_k = X_{(k)}$ as the sum of some of the U_i's. Find the $E(Y_k)$.

4. Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$ random variables. Define

$$Y_1 = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n), Y_2 = \frac{1}{\sqrt{2}}(X_1 - X_2)$$

$$Y_3 = \frac{1}{\sqrt{6}}(X_1 + X_2 - 2X_3), Y_4 = \frac{1}{\sqrt{12}}(X_1 + X_2 + X_3 - 3X_4)$$

(a) (7pts) Find the distribution for each of Y_1, Y_2, Y_3 and Y_4.

(b) (7pts) Given that Y_1, Y_2, Y_3, Y_4 are jointly normally distributed, we can prove that they are mutually independent by showing the pairwise correlation coefficient is zero. Show that $Cov(Y_1, Y_2) = 0$.

(c) (8pts) What is the distribution of $W = (Y_1 - \sqrt{n}\mu)/\sqrt{Y_2^2 + Y_3^2 + Y_4^2}$?

(d) (8pts) What is the distribution of $Y_2^2/(Y_3^2 + Y_4^2)$?